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A Linear Model Approach to Regression Analysis and its Applications

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The book develops the classical and modern unified approaches of linear models providing passage to Regression Analysis. While the theory of linear models have been treated exhaustively both by classical as well as by g-inverse approaches, the topic of Regression Analysis has ben presented through examples. The write-up of the book has been selective to cover the course within the time constraint of a semester; unlike other exhaustive authoritative treatises on the subject which besides being less comprehensive do not meet the requirement of doing justice to the subject within the time constraint of a semester. Derivation of many results of least squares theory by the use of g-inverse and Projection matrices, development of the background of ANOVA and ANCOVA through linear estimation approach and learning process of Regression Analysis through a complete chapter of examples are some of the unique features of the book.

Salient Features

- Fundamentals of Linear Estimation
- Testing of Linear Hypothesis
- Theory of Linear Estimation Based on g-inverse
- Miscellaneous Problems in Least Squares Theory
- Regression Theory and its Applications
- Application of Linear Estimation Techniques
- Problems and Solutions

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A LINEAR MODEL APPROACH
TO

REGRESSION ANALYSIS

AND ITS APPLICATIONS

SUDDHENDU BISWAS
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Delhi

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PREFACE

Regression Analysis is an important branch of Applied Statistics dealing with Multifactor data. The beauty of the subject lies in developing a conceptually simple model expressing the relationship between a set of meaningful variables. Of late, the methodology of the subject has undergone considerable innovation by the application of generalized inverse(especially Moore and Penrose Generalized inverse). Nevertheless, successful application of Regression Analysis requires an appreciation both theoretical as well as practical problems dealing with live data.

The main motivation of presenting this monograph is to fill up the need of the average graduate student pursuing a course in statistics covering this paper during a Semester (or a Trimester). There are books of highly exhaustive nature (like Draper & Smith Applied Regression Analysis, John Wiley & Sons) which may not exactly meet the requirements of students doing justice to the subject within the time constraint. In the second place, readability and understanding from some of the advanced texts in the subject is difficult for an average graduate student. Finally, a monograph like text dealing with the subject with major emphasis on generalized inverse following the line of approach of Linear Estimation Theory of late Prof. R. C. Bose (example, Derivation of Normal Equations of Gauss Markov model using projections and by M.P. g-inverse) fills up a considerable gap in the present literature. Hence my endeavour for a readable book for the graduate and Post-graduate students in statistics.

The first two chapters of the text are devoted to the development of linear Models and Testing of Linear Hypothesis in the classical manner; whereas, the third chapter is again a presentation of the Modelling and Analysis using generalized Inverse technique. The elementary basic results of g-inverse theory are already appended in the same chapter. Using the same techniques of solving most of the problems in Regression Analysis with ample illustrations a number of

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solved problems have been included in chapter four to develop insight of the learner of the subject. Chapter five deals with the concept of regression and develops regression analysis using the principles of least squares in linear models. Chapter six covers the techniques of ANOVA & ANCOVA using linear estimation theory. The last chapter is entirely devoted to the solution of all fundamental problems of linear regression, again providing an alternative approach for easy comprehensibility of the students.

I am thankful to my students, especially, Dr. Mariamma Thomas, Dr. Shisir Kumar Jha and Dr. (Mrs.) Sunita Chitkara and Dr. Saroj Kumar Adhikari for helping me in several ways while preparing the text. I recall a past student of mine (Ms. Shraddha Ghatpande in the University of Poona during 1972–74) who prepared notes delivered by me in her pearl like handwriting and gifted me with the same; which has become a basic ingredient for the write-up of the text. I am not aware of her present whereabouts but she remains ever in my memory.

My appreciation to my daughter Swati who helped me in several computations in the text.

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1st February' 95
Department of Statistics
Faculty of Mathematical Sciences
University of Delhi

ACKNOWLEDGEMENT

Dedicated to my Mentor
Late Professor P.K. Bose
Pro-Vice Chancellor and Head of the
Department of Statistics, Calcutta University

"A teacher can never truely teach
unless he is still learning himself
A lamp can never light another lamp
unless it continues to burn its own flame"

— *Rabindranath Tagore*

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Glossaries of Abbreviations and Notations

$\in$	≡	such that
$\exists$	≡	there exists
$D(Y)$	≡	Variance-Covariance Matrix of random Vector Y
$\rho(A)$	≡	Rank of the matrix A
G.M.	≡	Generalised Inverse
M.P.I.	≡	Moore and Penrose Inverse
G.M. set up	≡	Gauss Markov set up
BLUE	≡	Best linear unbiased estimator
$\hat{\theta}$	≡	Estimate of θ
$\perp^r$	≡	orthogonal
S.E.	≡	standard error
ANOVA	≡	Analysis of Variance
ANCOVA	≡	Analysis of Covariance
SS	≡	Sum of squares
OLSE	≡	ordinary least squares estimator
SSW	≡	Sum of squares within group
SSR	≡	Sum of squares due to regressions
GLSE	≡	Generalised least squares estimator

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FUNDAMENTALS OF LINEAR ESTIMATION

The motivation in this book is to develop the inference of linear stochastic models using the tools of Algebra of Matrices; especially using the generalised inverse theory (or in particular Moore and Penrose 'g' inverse theory which is unique).

The classical theory relating to the estimation of the parameters pertaining to a linear model orginated as early as Gauss (1809) followed by Markoff (1909). The result in systematic form, when the observations are known to have a given dispersion matrix, is classically known as Gauss–Markov theorem. However, the result was extended later on by Aitken (1935) Bose R.C. (1950 - 51), Neyman–David (1938) and Parzen(1951). Nevertheless, the problem remained unsolved, when the observations follow a dispersion matrix which is singular. As such the motivation of introducing results of g-inverse, especially the Moore and Penrose g-inverse was strong. The main contribution to the subject is due to Rao and Mitra (1968), Siskind and Martin (1969) etc. followed by their contribution to g-inverse theory which originated from Moore (1920), Penrose (1955), Bjerhammer (1958) and later on by Boullion and Odell (1971) and Chakrabarti (1970).

Unfied approaches comprise of the analysis of linear models which do not impose any restriction on the nature of the dispersion matrix, i.e.

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nonsingularity or so. This has been developed by Rao and Mitra (1971), Boullion and Odell (1971).

Another methodology relates to the estimation of Intra class correlation coefficients in a linear model for a two way clustered balanced data, again, when generalised to any set up, say, nonorthogonal nonconnected designs gave rise to **MINIQUE** Theory (Minimum Norm Quadratic Unbiased Estimation developed by Rao and also by Chakrabarti (1970) in respect of general analysis of designs.

1.1 LINEAR MODEL

A single equation linear model is given by

$$(Y)_{n\times 1} = (A)_{n\times m}\,(\Theta)_{m\times 1} + (U)_{n\times 1} \tag{1.1}$$

where $A = [a_{ij}]_{n\times m}$ is a matrix of known elements and $U = (u_1\ u_2\ \ldots\ u_n)'$ is the error column vector.

In full form it can be written as

$$\begin{aligned} y_1 &= a_{11}\,\theta_1 + a_{12}\,\theta_2 + \ldots + a_{1m}\,\theta_m + u_1 \\ y_2 &= a_{21}\,\theta_1 + a_{22}\,\theta_2 + \ldots + a_{2m}\,\theta_m + u_2 \\ &\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ y_n &= a_{n1}\,\theta_1 + a_{n2}\,\theta_2 + \ldots + a_{nm}\,\theta_m + u_n \end{aligned} \tag{1.2}$$

We further assume

$$E(Y) = A\Theta$$

$$\Rightarrow \quad E(u_i) = 0,\ \forall\ i = 1, 2, \ldots, n \tag{1.3}$$

$$D(U) = \sigma^2 I_n \tag{1.4}$$

i.e. variance covariance matrix of U is $\sigma^2 I_n$ and

$$\text{Cov}\,(u_i\ u_j) = 0 \tag{1.5}$$

i.e. the error components are uncorrelated (which becomes independent of u_i's are normally distributed). This is known as **Gauss Markov (G.M.) set up** $(Y, A\Theta, \sigma^2 I_n)$. An estimate of Θ under G.M. set up can be obtained by minimising

$$\begin{aligned} Q &= (Y - A\Theta)'\ (Y - A\Theta) \\ &= Y'Y - \Theta'A'Y - Y'A\Theta + \Theta'A'A\Theta \\ &= Y'Y - 2\Theta'A'Y + \Theta'A'A\Theta \end{aligned}$$

$$\delta Q/\delta\theta = 0 \ \Rightarrow\ -2A'Y + 2A'A\Theta = 0 \tag{1.6}$$

(by the rule of vector differentiation)

$$\Rightarrow \quad A'A\Theta = A'Y \text{ (known as normal equation)}$$

$$\Rightarrow \quad \hat{\Theta} = (A'A)^{-1}\,A'Y$$

is called the ***least squares estimate*** of Θ under G.M. set up. (C.R. Rao 1983).

1.2 GAUSS–AITKEN SET UP

A more generalised set up due to Aitken is given by

$$E(Y) = A(\Theta) \ \text{ and } \ D(U) = V \tag{1.8}$$

where D stands for dispersion matrix.

Now choose an orthogonal matrix P ∋

$$PVP' = \Lambda \tag{1.9}$$

where Λ is a diagonal matrix since V is positively definite. Premultiplying the linear model (1.1) by $\Lambda^{-1/2}P$, the same is transformed into

$$(\Lambda^{-1/2}P)\,Y = (\Lambda^{-1/2}P)\,A\Theta + \Lambda^{-1/2}PU \tag{1.10}$$

Now $D\,(\Lambda^{-1/2}PU) = \Lambda^{-1/2}P\,[D\,(U)]\,P'\,\Lambda^{-1/2}$

$$= \Lambda^{-1/2}PVP'\,\Lambda^{-1/2}$$

$$= \Lambda^{-1/2}\Lambda\ \Lambda^{-1/2} = I \tag{1.11}$$

Thus in the transformed model (1.10) since errors are uncorrelated [because of (1.11)], it is brought to G.M. set up and the least squares estimate of Θ, by replacing A by $(\Lambda^{-1/2}PA)$ and Y to $(\Lambda^{-1/2}PY)$ in (1.7), is given by

$$\begin{aligned} \hat{\Theta} &= [(\Lambda^{-1/2}PA)'\,(\Lambda^{-1/2}PA)]^{-1}\,(\Lambda^{-1/2}PA)'\,\Lambda^{-1/2}PY \\ &= [A'P'\,\Lambda^{-1/2}\ \Lambda^{-1/2}PA]^{-1}\,(A'P'\,\Lambda^{-1/2}\ \Lambda^{-1/2}PY) \\ &= [A'P'\,\Lambda^{-1}\,PA]^{-1}\,(A'P'\,\Lambda^{-1}PY) \end{aligned} \tag{1.12}$$

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Note : We have $PVP' = \Lambda$, but $P' = P^{-1}$; P being orthogonal

$$\Rightarrow \quad PV^{-1}P^{-1} = \Lambda^{-1} \Rightarrow P^{-1}\Lambda^{-1}P = V^{-1} \Rightarrow P'\Lambda^{-1}P = V^{-1}$$

$$\therefore \text{ (1.12) becomes } \hat{\Theta} = [A'V^{-1}A]^{-1}\ [A'V^{-1}Y] \tag{1.13}$$

This is called the **generalised least squares estimate (GLSE), $\hat{\Theta}$** or **Gauss–Aitken estimate** of Θ. Again when V is a singular matrix result (1.13) is not valid. It will be considered later on.

Also in case of

$$V = \sigma^2 I,$$

$$D(Y) = E[(Y - A\Theta)\ (Y - A\Theta)']$$

$$= E(VV') - E(V)\ E(V')$$

$$= \sigma^2 I$$

$$D\hat{\Theta} = D[(A'A)^{-1}A'Y]$$

$$= [(A'A)^{-1}A'D(Y)\ A\ (A'A)^{-1}]$$

$$= (A'A)^{-1}A'\sigma^2 IA\ (A'A)^{-1}$$

$$= \sigma^2 (A'A)^{-1}(A'A)(A'A)^{-1}$$

$$= \sigma^2 (A'A)^{-1} \tag{1.14}$$

$[(A'A)^{-1}$ being a symmetric matrix.]

Also in the general case of GLSE

$$\tilde{\Theta} = E[(Y - A\Theta)\ (Y - A\Theta)'] = V$$

$$\therefore \quad D(\tilde{\Theta}) = D[(A'V^{-1}A)^{-1}A'V^{-1}Y]$$

$$= [(A'V^{-1}A)^{-1}A'V^{-1}[D(Y)]\ V^{-1}A\ (A'V^{-1}A)^{-1}$$

$$= (A'V^{-1}A)^{-1}A'V^{-1}VV^{-1}A\ (A'V^{-1}A)^{-1}$$

$$= (A'V^{-1}A)^{-1}(A'V^{-1}A)\ (A'V^{-1}A)^{-1}$$

$$= (A'V^{-1}A)^{-1} \tag{1.15}$$

$[(A'V^{-1}A)$ being a symmetric matrix]

1.3 ESTIMABLE FUNCTIONS

A linear function $l'Y$ is estimable for a parametric function $l'\Theta$ if

$$E(l'Y) = b'\Theta \tag{1.16}$$

$$\Rightarrow \quad l'A\Theta = b'\Theta, \ \forall\ \Theta$$

$$\Rightarrow \quad l'A' = b'$$

$$\Rightarrow \quad (A)_{n\times m}(l)_{m\times 1} = b$$

The system of equations in $(l_1\ l_2 \ldots\ l_m)$ is consistent if

$$\rho(A) = \rho(A, b) \tag{1.17}$$

where $\rho \equiv$ Rank of the Matrix. Sometimes this is replaced by an alternative condition

$$\rho(A'A, b) = \rho(A'A) \tag{1.18}$$

A proof of this alternative condition is as follows :

The condition is Necessary

$$\rho(A'A) \le \rho(A'A, b)$$

$$\Rightarrow \quad \rho(A'A, b) \le \rho(A', b) = \rho(A') = \rho(A'A)$$

Thus $\quad \rho(A'A) \le \rho(A'A, b) \le \rho(A'A)$

$$\Rightarrow \quad \rho(A'A, b) = \rho(A'A) \tag{1.19}$$

The condition is Sufficient

Assume $\quad \rho(A'A) = \rho(A'A, b)$

$\Rightarrow \exists$ an equation $\quad (A'A)Z = b$ is consistent in Z.

Put $\quad AZ = Y \Rightarrow A'Y = b$ is consistent

$$\Rightarrow \quad \rho(A') = \rho(A', b)$$

which is also the condition of estimability.

Example 1.1 : $Y_1 = \mu + \beta_1 + u_1$; $Y_2 = \mu + \beta_2 + u_2$. Is $\beta_1 - \beta_2$ estimable under the G.M. set up ?

[**Ans** : Yes; because $E(Y_1 - Y_2) = \beta_1 - \beta_2$].

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1.3.1 Best Linear Unbiased Estimator (BLUE)

If $b'\Theta$ is estimable, then $\exists$ a linear function $C'Y \ni$

$$E(C'Y) = b'\Theta$$

$$\Rightarrow \quad C'A\Theta = b'\Theta \text{ holding } \forall\, \Theta$$

$$\Rightarrow \quad C'A = b'$$

$$\Rightarrow \quad A'C = b$$

$$\text{Var}(C'Y) = C_1^2 \text{Var}(Y_1) + C_2^2 \text{Var}(Y_2) + \ldots + C_n^2 \text{Var}(Y_n)$$

$$= C_1^2\sigma^2 + C_2^2\sigma^2 + \ldots + C_n^2\sigma^2$$

$$= \sum_{i=1}^{n} C_i^2\sigma^2 = C'C\sigma^2] \qquad (1.20)$$

Now in the class of all unbiased estimators our object is to find a linear function which has the minimum sampling variance for $b'\Theta$. Such a linear function is called the **Best Linear Unbiased Estimator (BLUE).**

1.3.2 Method of Construction of BLUE

If follows that we require to minimise $C'C$ subject to

$$A'C = b \qquad (1.21)$$

Thus, our problem reduces to unconditional minimisation of

$$f = C'C - 2\lambda'[A'C - b] \qquad (1.22)$$

where λ' is the Lagrangian multiplier. Differentiating f w.r.t. C, we have

$$\delta f/\delta C = 2C' - 2\lambda' A'$$

or

$$C = A\lambda \qquad (1.23)$$

Substituting (1.23) in (1.21) we get

$$A'A\lambda = b \qquad (1.24)$$

Note that the equation (1.23) is consistent as

$$\rho(A'A) = \rho(A'A,\ b)$$

$$\therefore \quad \lambda = (A'A)^{-1}b \qquad (1.25)$$

Substituting in equation (1.22) we have

$$C = A(A'A)^{-1}b \qquad (1.26)$$

provided $(A'A)^{-1}$ exists. Further, if $(A'A)$ is nonsingular, then from (1.26) we have

$$C'C\sigma^2 = b'(A'A)^{-1}A'A(A'A)^{-1}b\sigma^2$$

$$= b'(A'A)^{-1}b\sigma^2 \qquad (1.27)$$

which is the variance of BLUE. But if $(A'A)$ is singular, then a solution of λ in (1.24) is given by

$$\lambda = (A'A)^{-}b \qquad (1.28)$$

where $(A'A)^{-}$ is a g-inverse of $A'A$ and $(A'A)^{-}$ need not be unique. But whatever may be the solution of λ we shall show that the solution of C will remain unique. To show this:

Let

$$A = [A_{1(n\times r)} \mid A_{2(n\times m-r)}]_{n\times m} \qquad (1.29)$$

and assume without loss of generality that the first r columns are linearly independent and $(m-r)$ columns are depending on the first r columns; then

$$\rho(A) = r \qquad (1.30)$$

Also note that $[(A_1'A_1)^{-1}]_{r\times r}$ a submatrix of A of rank r exists. Let $\lambda^{(1)}$ and $\lambda^{(2)}$ be two distinct solutions of (1.24)

i.e. $\quad A'A\lambda^{(1)} = b$ and $A'A\lambda^{(2)} = b$

$$\Rightarrow \quad A'A(\lambda^{(1)} - \lambda^{(2)}) = 0 \qquad (1.31)$$

Consider $[C^{(1)} - C^{(2)}]'[C^{(1)} - C^{(2)}] \geq Q$

where $\quad C^{(1)} = A\lambda^{(1)}$ and $C^{(2)} = A\lambda^{(2)}$

$$\therefore \quad Q = [A\lambda^{(1)} - A\lambda^{(2)}]'[A\lambda^{(1)} - A\lambda^{(2)}]$$

$$= [\lambda^{(1)} - \lambda^{(2)}]'A'A[\lambda^{(1)} - \lambda^{(2)}] \qquad (1.32)$$

$$= 0, \text{ by (1.31)}$$

$$\Rightarrow \quad Q = [C^{(1)} - C^{(2)}]'[C^{(1)} - C^{(2)}] = 0$$

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$$C^{(1)} = C^{(2)}$$

But in general,

$$Q = [C^{(1)} - C^{(2)}]' \; [C^{(1)} - C^{(2)}] \geq 0 \qquad (1.33)$$

and the equality holds when $C^{(1)} = C^{(2)}$

Hence, with different solutions of λ the value of C remains unchanged or invariant.

Now $A'A\lambda = b \Rightarrow \lambda = (A'A)^{-}b$

A g-inverse of $A'A$ is given by

$$\begin{pmatrix} (A_1'A_1)^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

Let
$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \\ \hline b_{r+1} \\ \vdots \\ b_m \end{bmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$\therefore$
$$\lambda = \begin{pmatrix} (A_1'A_1)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$= \begin{pmatrix} (A_1'A_1)^{-1} b_1 \\ 0 \end{pmatrix}$$

Now $C = A\lambda$ gives that

$$= (A_1, A_2) \begin{pmatrix} (A_1'A_1)^{-1} b_1 \\ 0 \end{pmatrix}$$

$\therefore$ $C = A_1 (A_1'A_1)^{-1} b_1 \qquad (1.34)$

Also Var $(C'Y) = C'C\sigma^2$

$$= b_1' (A_1'A_1)^{-1} A_1'A_1 (A_1'A_1)^{-1} b_1 \sigma^2$$

$$= b_1' (A_1'A_1)^{-1} b_1 \sigma^2 \qquad (1.35)$$

Now we require to show that for any b

$$C'C \geq b_1' (A_1'A_1)^{-1} b_1 \qquad (1.36)$$

where C remains unique i.e. the variance of $C'Y$ given by (1.35) is minimum in the class of all unbiased estimates. Identically we have,

$$C'C = C'C - b_1' (A_1'A_1)^{-1} b_1 + b_1' (A_1'A_1)^{-1} b_1 \qquad (1.37)$$

In order that $C'Y$ is unbiased for $b'\Theta$ we have

$$A'C = b$$

$\Rightarrow$
$$\begin{pmatrix} A_1' \\ A_2' \end{pmatrix} C = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$\Rightarrow$ $A_1'C = b_1$ and $A_2'C = b_2 \qquad (1.38)$

Also $C'A = b'$

$\Rightarrow$ $C'(A_1, A_2) = (b_1', b_2')$

$\Rightarrow$ $C'A_1 = b_1' \qquad (1.39)$

Putting (1.38) and (1.39) in (1.37), we get

$$C'C = C'C - C'A_1 (A_1'A_1)^{-1} A_1'C + C'A_1 (A_1'A_1)^{-1} A_1'C$$

$$= C' [I - A_1 (A_1'A_1)^{-1} A_1'] C + b_1' (A_1'A_1)^{-1} b_1 \qquad (1.40)$$

Now $V \equiv [I - A_1 (A_1'A_1)^{-1} A_1']$

is symmetric and idempotent

$\therefore$ $C'C = (VC)'(VC) + b_1' (A_1'A_1)^{-1} b_1$

$\Rightarrow$ $C'C \geq b_1' (A_1'A_1)^{-1} b_1, \; \forall \; C$

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as $(VC)'(VC) \geq 0$

Hence, variance of BLUE for an estimable parametric function $b'\Theta$ is given by

$$b_1'(A_1'A_1)^{-1} b_1 \sigma^2 \tag{1.41}$$

when $(A'A)$ is singular.

Corollary:

Variance of blue of Θ is given by

$$D(\hat{\Theta}) = (A_1'A_1)^{-1} \sigma^2 \tag{1.42}$$

where $A = [\alpha_1, \alpha_2, ..., \alpha_m]$,

$$\alpha_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix} \text{ and } (A'A) \text{ is singular.}$$

1.4 ESTIMATION SPACE AND ERROR SPACE

The vector space spanned by $\alpha_1, \alpha_2 ..., \alpha_m$ is denoted by $V_\alpha (\alpha_1\ \alpha_2\ ...\ \alpha_m)$ and is known as the ***estimation space.*** A linear function $l'Y$ is said to belong to ***error space*** if

$$E(l'Y) = 0\ \forall\ \Theta$$

i.e. $$l'A\Theta = 0\ \forall\ \Theta$$

$\Rightarrow$ $$l'A = 0$$

Conversely, $l'A = 0 \Rightarrow l'A\Theta = 0, \forall\ \Theta$

i.e. $l'E(Y) = 0$ and $l'Y$ belongs to the ***error space.***

$\therefore$ the condition $l'A = 0$ is sufficient for a linear function $l'Y$ to belong to the error space.

1.5 NORMAL EQUATIONS

Under G.M. set up $E(Y) = A\Theta$, $(Y - A\hat{\Theta})$ will be called the **residual. The Residual Sum of Squares (RSS)** is given by

$$(Y - A\hat{\Theta})'(Y - A\hat{\Theta}) = Q, \text{ say}$$

$$= (Y' - \hat{\Theta}'A')(Y - A\hat{\Theta})$$

$$= (Y'Y - \hat{\Theta}'A'Y - Y'A\hat{\Theta} + \hat{\Theta}'A'A\hat{\Theta})$$

$$\delta Q/\delta\theta = 0$$

$\Rightarrow$ $$-2A'Y + 2A'A\hat{\Theta} = 0$$

$\Rightarrow$ $$\hat{\Theta} = (A'A)^{-1} A'Y \tag{1.43}$$

provided $A'A$ is nonsingular. Normal equations are given by

$$A'Y = A'A\hat{\Theta} \tag{1.44}$$

1.5.1 A Theorem on Normal Equations

Normal equations are always consistent.

Proof :

Normal equations $A'Y = A'A\hat{\Theta}$ are consistent; because

$$\rho(A'A) = \rho(A', A'Y) \tag{1.45}$$

Since $A'Y$ is a linear combination of vectors of A, (*i.e.* $\alpha_1, \alpha_2, ..., \alpha_m$) the normal equations are always solvable.

Remarks: However, if $(A'A)$ is singular, the solution of $\hat{\Theta}$ is not unique. But whatever solution of $\hat{\Theta}$ is taken $A\hat{\Theta}$ will remain unique.

To show this; let $\hat{\Theta}^{(1)}$ and $\hat{\Theta}^{(2)}$ be two solutions of $\hat{\Theta}$; then

$$A'Y = A'A\hat{\Theta}^{(1)} \text{ and } A'Y = A'A\hat{\Theta}^{(2)}$$

$\Rightarrow$ $$A'A(\hat{\Theta}^{(1)} - \hat{\Theta}^{(2)}) = 0 \tag{1.46}$$

Next consider

$$(A\hat{\Theta}^{(1)} - A\hat{\Theta}^{(2)})'(A\hat{\Theta}^{(1)} - A\hat{\Theta}^{(2)}) \geq 0$$

In view of equation (1.46)

$$[(\hat{\Theta}^{(1)} - \hat{\Theta}^{(2)})'A'A(\hat{\Theta}^{(1)} - \hat{\Theta}^{(2)}] = 0$$

$\Rightarrow$ $$A\hat{\Theta}^{(1)} = A\hat{\Theta}^{(2)}.$$

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1.6 GAUSS–MARKOV THEOREM

(*i*) Under G.M. set up, if $C'Y$ is BLUE for $b'\Theta$, then

$$C'Y = b'\hat{\Theta}$$

where $\hat{\Theta}$ is O.L.S.E. (Ordinary Least Squares Estimator).

(*ii*) $E[(Y - A\hat{\Theta})'(Y - A\hat{\Theta})] = (n-r)\sigma^2 = \text{R.S.S.}$

Proof (*i*) :

Case I : If $C'Y$ is unbiased for $b'\Theta$, then

$$E(C'Y) = b'\Theta$$

$$\Rightarrow \quad C'A\Theta = b'\Theta, \ \forall\ \Theta$$

$$\Rightarrow \quad A'C = b$$

Let $C = A\lambda$ be the consistent solution of $A'C = b$

$$\Rightarrow \quad C'Y = \lambda' A'Y \qquad (1.47)$$

Also $A'Y = A'A\Theta$ are the Normal Equations and the O.L.S.E. for Θ, say $\hat{\Theta}$ satisfies

$$A'Y = A'A\hat{\Theta}$$

$$\therefore \quad C'Y = \lambda' A'Y = \lambda' A'A\hat{\Theta}$$

$$A'C = A'(A\lambda) = b,$$

or $$b' = \lambda' A'A$$

$$\Rightarrow \quad b'\hat{\Theta} = \lambda' A'A\hat{\Theta}$$

$$\therefore \quad C'Y = \lambda' A'Y = \lambda' A'A\hat{\Theta} = b'\hat{\Theta}$$

[Note that the relation $C'Y = b'\hat{\Theta}$ considerably simplifies the computation of BLUE of a linear parametric function].

With $C'Y = b'\hat{\Theta}$

$$E(b'\hat{\Theta}) = E(C'Y) = C'A\Theta = \lambda' A'A\Theta \qquad (\text{as } C = A\lambda)$$

$$\Rightarrow \quad b'\Theta = C'A\Theta = \lambda' A'A\Theta$$

$$\therefore \quad b' = C'A = \lambda' A'A$$

$$b'\hat{\Theta} = \lambda' A'A\hat{\Theta}$$

$$E(b'\hat{\Theta}) = E(\lambda' A'Y) = \lambda' A'A\Theta = b'\Theta$$

which proves that $b'\hat{\Theta}$ is unbiased for $b'\Theta$

Also $\quad \text{Var}(b'\hat{\Theta}) = \text{Var}(C'Y)$

$$= C'C\sigma^2 I$$

$$= \lambda' A'A\lambda\sigma^2 \quad (\text{as } C = A\lambda)$$

$$= b'\lambda\sigma^2$$

$$\therefore \quad \text{VAr}(b'\hat{\Theta}) = b'(A'A)^{-1}\sigma^2;$$

if $A'A$ is nonsingular. Again from (1.27) variance of BLUE of an estimable parametric function is given by $b'(A'A)^{-1}\sigma^2$, if $A'A$ is nonsingular.

$$\Rightarrow \quad \text{Var}(b'\hat{\Theta}) = \text{Var}(\text{BLUE}) = \text{Var}(C'Y)$$

Case II : Also, if $A'A$ is singular we can write

$$\text{Var}(b'\hat{\Theta}) = b'(A'A)^{-}b\sigma^2$$

$$= (b_1' \ b_2') \begin{pmatrix} (A_1'A_1)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \sigma^2$$

$$\text{Var}(b'\hat{\Theta}) = b_1'(A_1'A_1)^{-1}b_1\sigma^2$$

Also, from (1.27) we see that variance of BLUE of an estimable parametric function is given by the same

i.e. $\quad b_1'(A_1'A_1)^{-1}b_1\sigma^2$

$\therefore\ b'\hat{\Theta}$ provides the BLUE.

Proof (*ii*) :

$$E[(Y - A\hat{\Theta})'(Y - A\hat{\Theta})] = (n-r)\sigma^2$$

We have, $\rho(A) = r$; $E(Y) = A\Theta$ and $D(Y) = \sigma^2 I$

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Normal equations are :

$$A'Y = A'A\,\hat{\Theta}$$

$$\Rightarrow \quad \hat{\Theta} = (A'A)^{-1} A'Y \text{ is not unique}$$

But as shown earlier,

$$A\,\hat{\Theta} = A\,(A'A)^{-}\,A'Y$$

remains invariant

$$\therefore \quad A\,\hat{\Theta} = (A_1, A_2) \begin{pmatrix} (A_1'A_1)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_1' \\ A_2' \end{pmatrix} Y$$

$$\Rightarrow \quad A\,\hat{\Theta} = A_1\,(A_1'A_1)^{-1}\,A_1'Y \qquad (1.48)$$

where $(A_1'A_1)$ is non-singular symmetric matrix. Hence, there exists a non-singular matrix G such that

$$G'G = (A_1'A_1)^{-1} \qquad (1.49)$$

(*G* can be obtained by Choleski's method of factorisation [Fröberg Carl E (1981).]

Now $A_1\,(A_1'A_1)^{-1}\,A_1' = A_1\,G'\,GA_1'$

$$= (GA_1')'\,(GA_1')$$

and $\rho\,[A_1\,(A_1'A_1)^{-1}\,A_1'] = \rho\,[A_1\,G'\,GA_1']$

$$= \rho\,[(GA_1')'\;GA_1')]$$

$$= \rho\,(GA_1')$$

$$= \rho\,(A_1') = r$$

If A is idempotent, then $(I-A)$ is also idempotent. Since, $A_1\,(A_1'A_1)^{-1}\,A_1'$ is symmetric and idempotent

trance $[A_1\,(A_1'A_1)^{-1}\,A_1'] = r$

$\therefore \quad \rho\,[I - A_1\,(A_1'A_1)^{-1}\,A_1'] = n - r$

Next, we require to show that

$$(Y - A\,\hat{\Theta}) = [I - A_1\,(A_1'A_1)^{-1}\,A_1']\,(Y - A\,\Theta)$$

$$= (Y - A\,\Theta) - A_1\,(A_1'A_1)^{-1}\,A_1'Y + A_1\,(A_1'A_1)^{-1}\,A_1'\,A\,\Theta \qquad (1.50)$$

Now $A\,\hat{\Theta} = A_1\,(A_1'A_1)^{-1}\,A_1'Y$, from (1.48)

Also $A_1\,(A_1'A_1)^{-1}\,A_1'A\,\Theta = A\,\Theta$

$\Rightarrow$ R.H.S. of (1.50) $= Y - A\,\Theta - A\,\hat{\Theta} + A\,\Theta$

$$= Y - A\,\hat{\Theta}$$

$$= \text{L.H.S. of (1.50)}$$

Again $(Y - A\,\hat{\Theta})'\;(Y - A\,\hat{\Theta})$

$$= (Y - A\,\Theta)'\,[I - A_1\,(A_1'A_1)^{-1}\,A_1']$$

$$[I - A_1\,(A_1'A_1)^{-1}\,A_1']\,(Y - A\,\Theta)$$

$$= (Y - A\,\Theta)'\,[I - A_1\,(A_1'A_1)^{-1}\,A_1']\,(Y - A\,\Theta) \qquad (1.51)$$

$\therefore \quad E\,[(Y - A\,\hat{\Theta})'\;(Y - A\,\hat{\Theta})]$

$$= E\,[(Y - A\,\Theta)'\,\{I - A_1\,(A_1'A_1)^{-1}\,A_1'\}\,(Y - A\,\Theta)]$$

R.H.S. of above being a quadratic form, i.e. a scalar, by putting

$[I - A_1\,(A_1'A_1)^{-1}\,A_1'] = V = [V_{ij}]$, we get

$$E\,[\Sigma\;\{y_i - E\,(y_i)\}^2\,V_{ii} + \sum_{i \neq j} E\,\{(y_1 - E\,(y_i)\,(y_j - E\,(y_j)\}\,V_{ij}]$$

second term of which is zero. Simplifying we have

$$\sigma^2 \sum_i V_{ii} = \sigma^2 \text{ trace } [I - A_1\,(A_1'A_1)^{-1}\,A_1']$$

$$= \sigma^2\,(n - r)$$

Hence, the estimate of σ^2 is given by

$$\hat{\sigma}^2 = \frac{(Y - A\,\hat{\Theta})'\;(Y - A\,\hat{\Theta})}{(n - r)} \qquad (1.52)$$

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$$\text{Var}(\hat{BLUE}) = b_1'(A_1'A_1)^{-1} b_1 \hat{\sigma}^2 = \frac{b_1'(A_1'A_1)^{-1} b_1 (Y - A\hat{\Theta})'(Y - A\hat{\Theta})}{(n-r)}$$

This proves Gauss–Markov theorem rigourously. It follows that σ^2 is estimated from

$$\frac{(Y - A\hat{\Theta})'(Y - A\hat{\Theta})}{(n-r)}$$

which is BLUE for σ^2, where $r = \rho(A)$.

Example 1.2 : Verify the following :

(i) $E(L'Y) = L'E(Y) = L'A\Theta$

(ii) $\text{Var}(L'Y) = L'D(Y)L$

$= L'\Sigma L$

$= L'\sigma^2 IL$; if $\Sigma = \sigma^2 I$

$= \sigma^2 L'L$

(iii) $\text{Cov}(L'Y, M'Y)$

$= E[\{L'Y - E(L'Y)\}(M'Y - E(M'Y)\}]'$

$- E[L'Y - E(L'Y)]\, E[M'Y - E(M'Y)]$

$= E[L'(Y - A\Theta)][M'(Y - A\dot{\Theta})]'$

$- E[L'(Y - A\Theta)][M'(Y - A\Theta)]$

$= L'E[(Y - A\Theta)(Y - A\Theta)']M$

$= L'\Sigma M$

$= L'\sigma^2 IM$; if $\Sigma = \sigma^2]$

$= \sigma^2 L'M$

(iv) If $(B)_{n\times k}(B'Y)_{k\times 1}$ is same as k linear functions of Y

$E(B'Y) = B'A\Theta$

$D(B'Y) = B'D(Y)B$

$= B'\Sigma B = \sigma^2 B'B$; where $\Sigma = \sigma^2 I$.

1.7 Distribution of Quadratic Forms

To derive the probability distribution of Quadratic forms $(X')_{1\times n}(A)_{n\times n}(X)_{n\times 1}$ where X is a random vector and A is a matrix of given elements with specified rank, say $\rho(A) = r \le n$, it is necessary to know certain Algebraic results. One such major result is known as Cochran's theorem which is stated as follows:

Cochran's Theorem

If $Q_i = X'A_iX$ $(i = 1, 2, \ldots, k)$ be k quadratic forms with $\rho(A_i) = r_i$ and if A_i's are positively semidefinite matrices,

i.e. $X'A_iX \ge 0$; $i = 1, 2, \ldots, k$ and if

$$Q = \sum_{i=1}^{n} Q_i = X'X = \sum_{\alpha=1}^{k} x_\alpha^2$$

then a necessary and sufficient condition that a common non-singular transformation, transforming each of

$Q_i = X'A_iX$; $(i = 1, 2, \ldots, k)$ to

$$Q_i' = \sum_{\alpha = r_1 + r_2 + \ldots + r_{i-1} + 1}^{r_1 + r_2 + \ldots + r_i} Z_\alpha^2$$

exists is that $r_1 + r_2 + \ldots + r_k = n$.

Proof : The Condition is Sufficient

We suppose that

$$r_1 + r_2 + \ldots + r_k = n$$

Since $\rho(A_i) = r_i$; $i = 1, 2, \ldots, k$, there exists a non-singular transformation э

$$P_i'A_iP_i = (I_{r_i}, 0)_{n\times n}$$

where $(I_{r_i}, 0) = \begin{pmatrix} I_{r_i} & 0 \\ 0 & 0 \end{pmatrix}$

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and $I_r = \text{diag}\,(1, 1 \ldots, 1);\ i = 1, 2, \ldots, k$

Next let us put

$$X = P_i Z \tag{1.53}$$

$$\Rightarrow \quad Q_1 = X' A_1 X = Z' P_1' A_1 P_1 Z$$

$$= Z' (I_{r_1}, 0)\, Z \tag{1.54}$$

$$= (z_1 z_2 \ldots z_n) \begin{bmatrix} I_{r_1}' & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

$$= \sum_{\alpha=1}^{r_1} z_\alpha^2$$

Similarly, $Q_2 = X' A_2 X = Z' P_2' A_2 P_2 Z$

$$= (z_1 z_2 \ldots z_n) \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & I_{r_2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

$$= \sum_{\alpha=r_1+1}^{r_1+r_2} z_\alpha^2$$

Similarly, $Q_3 = \sum_{\alpha=r_1+r_2+1}^{r_1+r_2+r_3} z_\alpha^2$

and finally, $Q_k = \sum_{\alpha=r_1+\ldots+r_{k-1}+1}^{r_1+\ldots+r_k} z_\alpha^2$

Hence, $Q = Q_1 + Q_2 + \ldots + Q_k$

$$= \sum_{\alpha=1}^{r_1+\ldots+r_k} z_\alpha^2$$

$$= \sum_{\alpha=1}^{n} z_\alpha^2$$

(by assumption $r_1 + r_2 + \ldots + r_k = n$)

Again, if $X = PZ$ is the transformation, transforming

$Q = X'X$ into

$$Z'P'PZ = \sum_{\alpha=1}^{n} z_\alpha^2$$

$$= Z'P'\, I_n\, PZ;\ \forall\ Z$$

Further,

$$Q_1 + Q_2 + \ldots + Q_k = Z' \left[(P_1' A_1 P_1) + (P_2' A_2 P_2) + \ldots + (P_k' A_k P_k)\right] Z$$

$$= Z'\, P'\, I_n\, PZ$$

which holds $\forall\ Z$

$$\Rightarrow \quad P'\, I_n P = P_1' A_1 P_1 + P_2' A_2 P_2 + \ldots + P_k' A_k P_k$$

$$= P' (A_1 + A_2 + \ldots + A_n)\, P$$

$$= P'\, P \left(\text{as} \sum_{i=1}^{n} A_i = I_n \right) \tag{1.55}$$

The result holds if $P_1 = P_2 = \ldots = P_k = P$.

Putting $P_i = P;\ (i = 1, 2, \ldots, k)$ in (1.53) and (1.54) and noting that if $r_1 + r_2 + \ldots + r_k = n$, then $\exists$ a common transformation $X = PZ$ which transforms

$$Q_1 = X' A_1 X = Z'\, P'\, A_1\, PZ$$

$$= Z' (I_{r_1}, 0)\, Z$$

$$= \sum_{\alpha=1}^{r_1} z_\alpha^2$$

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$$Q_2 = X' A_2 X = Z' P' A_2 PZ$$
$$= Z' (I_{r_2}, 0) Z$$
$$= \sum_{\alpha = r_1 + 1}^{r_1 + r_2} z_\alpha^2$$

and
$$Q_k = X' A_k X = Z' P' A_k PZ$$
$$= Z' (I_{r_k}, 0) Z$$
$$= \sum_{\alpha = r_1 + \ldots + r_{k-1} + 1}^{r_1 + \ldots + r_k} z_\alpha^2$$

The condition is Necessary

We have

$$Q = \sum_{\alpha = 1}^{r_1 + \ldots + r_k} z_\alpha^2 = \sum_{\alpha = 1}^{n} z_\alpha^2 = X' X$$

under a common non-singular transformation $X = PZ$

If possible, let $r_1 + r_2 + \ldots + r_k < n$, then

$$\rho (Q) < n$$

which is contrary to our hypothesis $\rho (Q) = n$. On the other hand if $r_1 + r_2 + \ldots + r_k > n$, then the transformation $X = PZ$ is singular which is again in contradiction to the hypothesis.

1.7.1 Statistical Translation of Cochran's Theorem

If $(X)_{n \times 1}$ is a random vector $\ni (X)_{n \times 1} \sim N (0, \sigma^2 I_n)$ and $Q = X'X \sim \chi_n^2$ (a chi square with n degrees of freedom), then a necessary and sufficient condition that each

$$Q_i = X' A_i X \sim \chi_{n_i}^2$$

(a chisquare with n_i degrees of freedom) with

$$\sum_{i=1}^{n} A_i = I_n, \text{ is that}$$

$$\sum_{i=1}^{n} \rho (A_i) = r_1 + r_2 + \ldots + r_k = n$$

1.7.2 Another Important Result

Theorem: A necessary and sufficient condition that $Q = X'AX$ can be expressed as r sum of squares, where $r = \rho (A)$, by an orthogonal transformation $X = PZ$ is that A is an idempotent matrix.

Proof : (*i*) The Condition is Necessary

If $X = PZ$ be an orthogonal transformation by which

$$Q = X'AX \rightarrow Z' P'APZ$$
$$= Z' P^{-1} APZ \; (P' = P^{-1} \text{ as } P \text{ is orthogonal})$$
$$= \sum_{\alpha = 1}^{r} z_\alpha^2$$

$$\Rightarrow \quad P^{-1} AP = (I_r, 0) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad A = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} P^{-1}$$

$$\Rightarrow \quad A^2 = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} P^{-1} P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} P^{-1}$$
$$= P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} P^{-1} = A$$

which shows A to be an idempotent matrix.

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(ii) The Condition is Sufficient

If A is an idempotent matrix with $r(A) = r$, then $\exists$ an orthogonal matrix P ∋

$$P^{-1}AP = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

Hence, under an orthogonal transformation $X = PZ$, $Q = X'AX$ transforms to

$$\begin{aligned} Q &= Z'P'APZ \\ &= Z'P^{-1}APZ \\ &= Z'\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}Z \\ &= \sum_{\alpha=1}^{r} z_\alpha^2 \end{aligned}$$

1.7.3 Statistical Interpretation of Result

If $$X \sim N(0, \sigma^2 I_n), \text{ then}$$

$$Q = X'AX \sim \chi_r^2$$

for $\rho(A) = r$ if A is an idempotent matrix.

Example 1.3 : If $(X_1, X_2 ..., X_k)'$ is a vector r.v. having k variate normal distribution $N(\Lambda, \Sigma)$, where $\Lambda' = (\mu_1, \mu_2 ..., \mu_k)$ and $\Sigma = [\sigma_{ij}]$; $i, j = 1, 2, ..., k$, then, $1/2\, Q(X_1, X_2, ..., X_k)$ is a Gamma distributed vector with parameter $k/2$ where

$$Q = (X_1 - \mu_1,\ X_2 - \mu_2, \ldots X_k - \mu_k)\,(\sigma_{ij})^{-1} \begin{pmatrix} X_1 - \mu_1 \\ X_2 - \mu_2 \\ \vdots \\ X_k - \mu_k \end{pmatrix}$$

$$= (X_1 - \mu_1,\ X_2 - \mu_2, \ldots X_k - \mu_k)\,(\sigma_{ij}) \begin{pmatrix} X_1 - \mu_1 \\ X_2 - \mu_2 \\ \vdots \\ X_k - \mu_k \end{pmatrix}$$

Proof : Consider the characteristic function

$$\begin{aligned} \phi &= 1/2\, Q(X_1, X_2, ..., X_k) \\ &= E\left(e^{itQ(X_1, X_2, \ldots X_k)/2}\right) \end{aligned}$$

$$\phi = \frac{\sqrt{|\sigma^{ij}|}}{(2\pi)\,(k/2)} \int \ldots \int_{R_k} e^{itQ/2}\, e^{-(Q/2)}\, dx_1\, dx_2 \ldots dx_k$$

where $[\sigma^{ij}] = [\sigma_{ij}]^{-1}$

Employing the transformation

$$Y_i = \sqrt{1 - it}\,(X_1 - \mu_1);\ i = 1, 2, ..., k$$

We get the characteristic function

$$\phi = \frac{\sqrt{|\sigma^{ij}|}}{(2\pi)\,(k/2)}\,(1 - it)^{-k/2} \int \ldots \int_{R_1} e^{-1/2 \sum \sigma_{ij}\, y_i \cdot y_j \cdot dy_1\, dy_2 \ldots dy_n}$$

$$= (1 - it)^{-k/2}$$

which is the characteristic function of a one parameter family of Γ distribution.

1.7.4 Expectation of a Quadratic Form

$$E(Y'GY) = E\left[\sum_{ij} y_i\, y_j\, g_{ij}\right]$$

$$\begin{aligned} &= E(y_1^2)\, g_{11} + E(y_1 y_2)\, g_{12} + E(y_1 y_3)\, g_{13} + \ldots + E(y_1 y_n)\, g_{1n} \\ &\quad + E(y_2 y_1)\, g_{21} + E(y_2^2)\, g_{22} + \ldots + E(y_2 y_n)\, g_{2n} + \ldots \\ &\quad + E(y_n y_1)\, g_{n1} + E(y_n y_2)\, g_{n2} + \ldots l + E(y_n^2)\, g_{nn} \end{aligned}$$

where $E(y_i) = \mu_i$ and $E(y_i y_j) - E(y_i)\, E(y_j) = \text{Cov}(y_i y_j) = \sigma_{ij}$

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$$\Rightarrow E(Y'GY) = (\sigma_{11} + \mu_1^2)\, g_{11} + (\sigma_{12} + \mu_1 \mu_2)\, g_{12}$$
$$+ (\sigma_{13} + \mu_1 \mu_3)\, g_{13} + \ldots + (\sigma_{1n} + \mu_1 \mu_n)\, g_{1n}$$
$$+ \ldots + (\sigma_{n1} + \mu_n \mu_1)\, g_{n1} + \ldots + (\sigma_{nn} + \mu_n^2)\, g_{nn}$$
$$= \Sigma\, \sigma_{1j}\, g_{1j} + \Sigma\, \sigma_{2j}\, g_{2j} + \ldots + \Sigma\, \sigma_{nj}\, g_{nj} + \mu_1^2\, g_{11}$$
$$+ 2\mu_1 \mu_2\, g_{12} + \ldots + 2\mu_1 \mu_n\, g_{1n} + \mu_2^2\, g_{22}$$
$$+ 2\mu_2 \mu_3\, g_{23} + \ldots + 2\mu_2 \mu_n\, g_{2n} + \ldots + \mu_n^2\, g_{nn}$$
$$= \text{trace}\; G\Sigma + (\mu_1 \ldots \mu_n)\,(g_{ij})\,(\mu_1 \ldots \mu_n)'$$
$$= tr\, G\Sigma + E(Y')\, GE(Y);$$

where $\Sigma = D(Y)$

$$= tr\,(G\sigma^2 + E(Y')\, GE(Y), \text{ if } \Sigma = \sigma^2]$$
$$= \sigma^2\, tr\, G + E(Y')\, GE(Y).$$

2

TESTING OF LINEAR HYPOTHESIS

2.1 FIRST PROBLEM

Testing H_0 : $\Theta = 0$ against H_1 : $\Theta \neq 0$

We have $(Y - A\Theta)'(Y - A\Theta)$

$$= [Y - A\hat{\Theta} + A(\hat{\Theta} - \Theta)]'\,[(Y - A\hat{\Theta} + A(\hat{\Theta} - \Theta)]$$
$$= (Y - A\hat{\Theta})'(Y - A\hat{\Theta}) + (\hat{\Theta} - \Theta)' A'(Y - A\hat{\Theta})$$
$$+ (Y - A\hat{\Theta})' A(\hat{\Theta} - \Theta) + (\hat{\Theta} - \Theta)' A'A(\hat{\Theta} - \Theta) \qquad (2.1)$$

Next consider

$$(Y - A\hat{\Theta})' A(\hat{\Theta} - \Theta) = Y'A\hat{\Theta} - \hat{\Theta}' A'A\hat{\Theta} - Y'A\Theta + \hat{\Theta}' A'A\Theta$$

Again $\quad A'Y = A'A\hat{\Theta} \Rightarrow Y'A\hat{\Theta} = \hat{\Theta} A'A\hat{\Theta}$

Also $\quad Y'A\Theta = \hat{\Theta} A'A\Theta$

$$\Rightarrow (Y - A\hat{\Theta})' A(\hat{\Theta} - \Theta)$$
$$= \hat{\Theta}' A'A\hat{\Theta} - \hat{\Theta}' A'A\hat{\Theta} - \hat{\Theta} A'A\Theta + \hat{\Theta} A'A\Theta = 0 \qquad (2.2)$$

Precisely in the same way,

$$(\hat{\Theta} - \Theta)' A'(Y - A\hat{\Theta}) = [(Y - A\hat{\Theta})' A(\hat{\Theta} - \Theta)] = 0 \qquad (2.3)$$

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Hence, from (2.1), (2.2) and (2.3) we have

$$(Y - A\Theta)'(Y - A\Theta) = (Y - A\hat{\Theta})'(Y - A\hat{\Theta}) + (\hat{\Theta} - \Theta)'A'A(\hat{\Theta} - \Theta) \quad (2.4)$$

Also $$(Y - A\Theta)'(Y - A\Theta) = (Y - A\hat{\Theta})'[I - A_1(A_1'A_1)^{-1}A_1'](Y - A\hat{\Theta}) \quad (2.5)$$

Hence, it follows that

$$(\hat{\Theta} - \Theta)'A'A(\hat{\Theta} - \Theta) = (Y - A\Theta)'A_1(A_1'A_1)^{-1}A_1'(Y - A\Theta) \quad (2.6)$$

Since R.H.S. of (2.6) + R.H.S. of (2.5) = L.H.S. of (2.4), we have

$$(Y - A\Theta)'(Y - A\Theta) = (Y - A\hat{\Theta})'[I - A_1(A_1'A_1)^{-1}A_1'](Y - A\hat{\Theta}) + (Y - A\hat{\Theta})'A_1(A_1'A_1)^{-1}A_1'(Y - A\hat{\Theta})$$

$$= \text{d.f. of } (Y - A\Theta)'(Y - A\Theta)$$

$$= (n - r) + r = n$$

Now, $E(Y - A\hat{\Theta}) = 0$ and $D(Y - A\hat{\Theta}) = \sigma^2 I$

$$\rho[I - A_1(A_1'A_1)^{-1}A_1'] + \rho[A_1(A_1'A_1)^{-1}A_1'] = (n - r) + r$$

$$= n = \rho(I_n)$$

and v and $(I - v)$ are independent, where $v = A_1(A_1'A_1)^{-1}A_1'$. Hence, by applying Cochran's theorem (vide 1.7.1), we have

$$(Y - A\hat{\Theta})'(Y - A\hat{\Theta}) \text{ and } (\hat{\Theta} - \Theta)'A'A(\hat{\Theta} - \Theta)$$

are independently distributed as $\chi^2\sigma^2$ with $(n - r)$ and r d.f., where

$$(\hat{\Theta} - \Theta)'A'A(\hat{\Theta} - \Theta)$$

is the regression sum of squares and

$$(Y - A\hat{\Theta})'(Y - A\hat{\Theta})$$

is the residual sum of squares under $H_0 : \Theta = 0$

$\therefore$ to test $H_0 : \Theta = 0$ against $\Theta \neq 0$

$$F_{r,n-r} = \frac{\hat{\Theta}'A'A\hat{\Theta}/r}{(Y - A\hat{\Theta})'(Y - A\hat{\Theta})/(n - r)} \quad (2.7)$$

which is distributed as Snedecor's F with r and $(n - r)$ degrees of freedom. Also,

$$(Y - A\hat{\Theta})'(Y - A\hat{\Theta}) = Y'Y - Y'A\hat{\Theta} - \hat{\Theta}'A'Y + \hat{\Theta}'A'A\hat{\Theta}$$

$$= Y'Y - \hat{\Theta}'A'A\hat{\Theta} - \hat{\Theta}'A'A\hat{\Theta} + \hat{\Theta}'A'A\hat{\Theta}$$

But $A'Y = A'A\hat{\Theta} \Rightarrow Y'A = \hat{\Theta}'A'A$

$\therefore$ $(Y - A\hat{\Theta})'(Y - A\hat{\Theta}) = Y'Y - \hat{\Theta}'A'A\hat{\Theta}$

$\Rightarrow$ $$F = \frac{\hat{\Theta}'A'A\hat{\Theta}/r}{Y'Y - \hat{\Theta}'A'A\hat{\Theta}/n - r} \quad (2.8)$$

or $$F = \frac{\hat{\Theta}'A'Y/r}{Y'Y - \hat{\Theta}'A'Y/n - r} \quad (2.9)$$

2.2 SECOND PROBLEM

To test $H_0 : b'\Theta = 0$ against $H_1 : b'\Theta \neq 0$

Let us define

$$Z = \begin{bmatrix} z_1 \\ \vdots \\ z_r \\ \hline z_{r+1} \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} (Z_1)_{r\times 1} \\ \\ (Z_2)_{n-r\times 1} \end{bmatrix}$$

We have earlier referred that $\exists$ a non-singular matrix $G \ni$ $(A_1'A_1)^{-1} = G'G$ [vide (1.49)]

Hence, $$(A_1'A_1)^{-1}G^{-1} = G' \quad (2.10)$$

We now show that $(GA_1')_{r\times n}$ is an array of r n-dimensional orthonormal vectors.

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$$(GA_1')\,(GA_1')' = GA_1'\,A_1\,G'$$

$$(GA_1')_{r\times n}\,(GA_1')'_{n\times r} = \frac{GA_1'\,A_1\,(A_1'\,A_1)^{-1}\,G^{-1}}{I_r} \qquad (2.11)$$

Let $\begin{bmatrix} (GA_1')_{r\times n} \\ (L)_{n-r\times n} \end{bmatrix}_{n\times n}$

be an n-square matrix where L represents a similar array like GA_1' of $(n-r)$ orthonormal vectors and L is arbitrary which can be chosen in any number of ways. Let us denote

$$(Z)_{n\times 1} = \begin{pmatrix} GA_1' \\ L \end{pmatrix}_{n\times n} (Y - A\hat{\Theta})_{n\times 1}$$

then $$Z'Z = (Y-A\Theta)'\,I_n\,(Y-A\Theta) = \sum_{i=1}^{n} z_i^2 \qquad (2.12)$$

Also, $(Y-A\Theta)'\,A_1\,(A_1'\,A_1)^{-1}\,A_1'\,(Y-A\Theta)$

$$= (Y-A\Theta)'\,A_1\,G'\,GA_1'\,(Y-A\Theta)$$

$$= [GA_1'\,(Y-A\Theta)]'\,[GA_1'\,(Y-A\Theta)]$$

$$= Z_1'\,Z_1 = \sum_{i=1}^{r} z_i^2 \qquad (2.13)$$

Finally,

$$(Y-A\Theta)'\,[I - A_1\,(A_1'\,A_1)^{-1}\,A_1']\,(Y-A\Theta) = Z_2'\,Z_2 \qquad (2.14)$$

[Because, LHS of (2.14) + LHS of (2.13) = LHS of (2.12) + 1

and RHS of (2.14)

= RHS of (2.12) – RHS of (2.13)]

$$Z_2'\,Z_2 = \sum_{i=r+1}^{n} z_i^2$$

thus $Z_i \sim N(0, \sigma^2)$;

since RHS of (2.12)$/\sigma^2 = \chi_n^2$;

RHS of (2.13)$/\sigma^2 = \chi_r^2$

and RHS of (2.14)$/\sigma^2 = \chi_{n-r}^2$

and all are independently distributed

$\therefore$ BLUE of $b'\,\hat{\Theta} = C'Y$

Also BLUE of

$$b'\,\hat{\Theta} = b_1\,(A_1'\,A_1)^{-1}\,A_1'Y$$

$$= b_1'\,(A_1'\,A_1)^{-1}\,A_1'\,(Y-A\Theta) + b_1'\,(A_1'\,A_1)^{-1}\,A_1'A\Theta$$

$$= b_1'\,G'\,GA_1'\,(Y-A\Theta) + C'A\Theta$$

(because $C' = b_1'\,(A_1'\,A_1)^{-1}\,A_1'$)

But $C'A = b'$ by the condition of estimability

$\therefore$ $$b'\,\hat{\Theta} = b'\,\Theta + b_1'\,G'\,GA_1'\,(Y-A\Theta)$$

$$= b'\Theta + b_1'\,G'Z_1 \qquad (2.15)$$

Distribution of $b'\,\hat{\Theta}$, which is linear combination of normal variables, is normal with mean zero

Now $-\,b_1'\,G'\,Z_1 = b'\,\Theta - b'\,\hat{\Theta}$

and under H_0, $b'\,\Theta$ must be zero

$\therefore$ $b_1'\,G'Z_1 = b'\,\hat{\Theta}$

$\therefore$ $C'Y = b'\,\hat{\Theta} = b_1'\,G'Z_1$

$$= [b_1'\,(A_1'\,A_1)^{-1}\,A_1'Y]$$

$\therefore$ Var $(C'Y)$ = Var $(b'\,\hat{\Theta})$

$$= b_1'\,(A_1'\,A_1)^{-1}\,b_1\,\sigma^2$$

Hence, $$\frac{b_1'\,(A_1'\,A_1)^{-1}\,A_1'\,Y}{\sqrt{b_1'\,(A_1'\,A_1)^{-1}\,b_1\,\sigma^2}} \qquad (2.16)$$

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is a standard normal variate. We have also shown that the residual mean $S.S \sim \sigma^2 \chi^2_{n-r}$

i.e. $$\frac{\text{Residual Mean S.S}}{\sigma^2} = \frac{Y'Y - (\hat{\Theta}' A'Y)}{\sigma^2 (n-r)}$$

$$\sim \chi^2_{n-r}$$

Hence, an appropriate test is

$$t = \frac{b_1'(A_1'A_1)^{-1}A_1'Y / b_1'(A_1'A_1)^{-1} b_1}{\sqrt{Y'Y - \hat{\Theta}'A'Y/(n-r)}}$$

where t is distributed as Student's t with $(n-r)$ d.f.

Example 2.1 : One manufacturing company of electric bulb while testing the life time of bulbs manufactured by them took a random sample of n bulbs from a manufactured lot. All the n bulbs were lighted for a period of t hours; and the distribution of $k < n$ bulbs which were exhausted during t hours are given in order as $t_{(1)} < t_{(2)} < \ldots < t_{(k)}$. Assuming that the life times of bulbs are distributed independently and exponentially given by $f(t) = \lambda e^{-\lambda t}$. Obtain the BLUE of the mean life time of the bulbs and the standard error of BLUE. 188625

Hints : Here, $t_{(1)} < t_{(2)} < \ldots < t_{(k)}$ are not independent. But $t_{(1)}$, $t_{(2)} - t_{(1)}$, ... $t_{(k)} - t_{(k-1)}$ are independent random variables whose distributions are given by

$$P(t_{(r)} - t_{(r-1)} > t) = e^{-\lambda (n-r+1) t}$$ [Biswas.S (1994)]

Now construct a linear function of of the form

$$L = \mu_1 t_{(1)} + \mu_2 (t_{(2)} - t_{(1)}) + \ldots + \mu_k (t_{(k)} - t_{(k-1)})$$

Set $E(L) = 1/\lambda$ (mean of an exponential distribution with parameter λ), subject to Var (L) to be minimum i.e. unconditionally minimise

$$Q = \text{Var}(L) - \alpha [E(L) - 1/\lambda]$$

where α is the Lagrangian Multiplier Differentiate w.r.t. $\mu_1, \mu_2, \ldots \mu_k$ as well as w.r.t α. We get BLUE and the standard error of BLUE.

Example 2.2 : $E(Y) = X\Theta$ where $Y = X\Theta + U$ under no specification on U, the error vector. Is Θ, the ordinary least squares estimator, unbiased?

2.3 THIRD PROBLEM

To test $H_0 : b_i'\Theta = 0$ against $H_1 : b_i'\Theta \neq 0$; $(i = 1, 2, \ldots, k)$ i.e. testing k linear parametric functions simultaneously

Suppose out of k linear parametric functions k' $(k' \leq k)$ are independent and the remaining $(k - k')$ functions are dependent on the first k' functions.

Let $$b_i = \begin{bmatrix} b_{i1} \\ b_{i2} \\ \vdots \\ b_{ir} \\ \hline b_{ir+1} \\ \vdots \\ b_{im} \end{bmatrix} = \begin{bmatrix} (b_{i1})_{r \times 1} \\ (b_{i2})_{m-r \times 1} \end{bmatrix}$$, say

BLUE of $b'\Theta = b_1'(A_1'A_1)^{-1}A_1'Y = C'Y$

BLUE of $b_1'\Theta = b_{11}'(A_1'A_1)^{-1}A_1'Y$

BLUE of $b_2'\Theta = b_{21}'(A_1'A_1)^{-1}A_1'Y$

BLUE of $b_k'\Theta = b_{k1}'(A_1'A_1)^{-1}A_1'Y$

$\therefore$ BLUE of all the k independent parametric functions is $B_1(A_1'A_1)^{-1}A_1'Y$, where

$$B_1 = \begin{bmatrix} b_{11}' \\ b_{21}' \\ \vdots \\ b_{k1}' \end{bmatrix}$$

Again we have,

$$\text{Var}(b'\hat{\Theta}) = b_1'(A_1'A_1)^{-1} b_1 \sigma^2$$

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$$\text{Var}(b_1'\hat{\Theta}) = \text{Var (BLUE of } b_1'\Theta) = b_{11}'(A_1'A_1)^{-1}b_{11}\sigma^2$$

$$\text{Var}(b_2'\hat{\Theta}) = \text{Var BLUE of } b_2'\Theta) = b_{21}'(A_1'A_1)^{-1}b_{21}\sigma^2$$

$$\vdots : \vdots : \vdots : \vdots : \vdots$$

$$\text{Var}(b_k'\hat{\Theta}) = \text{Var (BLUE of } b_k'\Theta) = b_{k1}'(A_1'A_1)^{-1}b_{k1}\sigma^2$$

Since, $b_1'\Theta,\ b_2'\Theta, \ldots, b_k'\Theta$ are all stochastically independent

$$\text{Cov}(b_i'\hat{\Theta}, b_j'\hat{\Theta}) = 0;\ \forall\ i, j = 1, 2, \ldots, k;\ i \neq j$$

Hence, the variance-covariance matrix of the BLUE of k' independent functions is given by

$$\sigma^2\begin{bmatrix} b_{11}'(A_1'A_1)^{-1}b_{11} & 0 & \cdots & 0 \\ 0 & b_{21}'(A_1'A_1)^{-1}b_{21} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{k1}'(A_1'A_1)^{-1}b_{k1} \end{bmatrix}$$

$$= \sigma^2\begin{bmatrix} b_{11}' \\ b_{21}' \\ \vdots \\ b_{k1}' \end{bmatrix}(A_1'A_1)^{-1}(b_{11}\ b_{21}\ \ldots\ b_{k1})$$

$$= \sigma^2 B_1(A_1'A_1)^{-1}B_1' \qquad (2.18)$$

Now $X = B_1(A_1'A_1)^{-1}A_1'Y$

Consider $Q = X'[D(X)]^{-1}X$

$$= Y'A_1(A_1'A_1)^{-1}B_1'[(B_1)(A_1'A_1)^{-1}B_1']^{-1}\sigma^{-2}$$

$$[B_1(A_1'A_1)^{-1}A_1'Y] \qquad (2.19)$$

Note that $[B_1(A_1'A_1)^{-1}B_1']$ is non-singular with rank k'. Under $H_0 : b_i'\Theta = 0;\ i = 1, 2, \ldots k$

$$\Rightarrow \qquad E(b_i'\Theta) = 0\ \forall\ i$$

[because $b'\hat{\Theta} = \text{BLUE of } b'\Theta = b_1'(A_1'A_1)^{-1}A_1'Y$]

$$\therefore \qquad E[b_{i1}'(A_1'A_1)^{-1}A_1'Y] = 0;\ \text{for } i = 1, 2, \ldots, k'$$

or
$$\begin{bmatrix} b_{11}' \\ b_{21}' \\ \vdots \\ b_{k1}' \end{bmatrix}(A_1'A_1)^{-1}A_1'A\Theta = 0 \qquad (2.20)$$

or
$$B_1(A_1'A_1)^{-1}A_1'A\Theta = 0 \qquad (2.21)$$

Taking transposed matrix of above, we have,

$$\Theta'A'A_1(A_1'A_1)^{-1}B_1' = 0 \qquad (2.22)$$

Multiplying $[B_1(A_1'A_1)^{-1}B_1']^{-1}$, which is non-singular with rank $= k'$, by (2.22) and postmultiplying by (2.21) we get

$$\Theta'A'A_1(A_1'A_1)^{-1}B_1'[B_1(A_1'A_1)^{-1}B_1']^{-1}$$

$$B_1(A_1'A_1)^{-1}A_1'A_1\Theta = 0 \qquad (2.23)$$

Now subtracting (2.23) from (2.19), we have

$$Q = (Y - A\Theta)'A_1(A_1'A_1)^{-1}B_1'[B_1(A_1'A_1)^{-1}B_1']^{-1}$$

$$\times B_1(A_1'A_1)^{-1}A_1'(Y - A\Theta)\sigma^{-2} \qquad (2.24)$$

[i.e. Y in (2.19) being replaced by $(Y - A\Theta)$]

If we denote as before, using (2.10) and (2.12)

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{pmatrix} GA_1' \\ L \end{pmatrix}(Y - A\Theta) = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$$

where $Z'Z = (Y - A\Theta)'(Y - A\Theta)$ with n d.f.

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$Z_1' Z_1 = (Y - A\Theta)' A_1 (A_1' A_1)^{-1} A_1' Y (Y - A\Theta)$ with r d.f.

$Z_2' Z_2 = (Y - A\Theta)' [I - A_1 (A_1' A_1)^{-1} A_1' Y] \ (Y - A\Theta)$

with $(n - r)$ d.f. and $(A_1' A_1)^{-1} = G'G$

In view of $GA_1' (Y - A\Theta) = Z_1$

$Z_1' = (Y - AQ)' A_1 G'$ and $(A_1' A_1)^{-1} = G'G$

$$\Rightarrow \quad Q = (Y - A\Theta)' A_1 \, G' GB_1' [B_1 GG' B_1']^{-1} B_1 \, G' GA_1' (Y - A\Theta)$$

$$= Z_1' [GB_1' [B_1 (A_1' A_1)^{-1} B_1']^{-1} B_1 \, G'] Z_1 \qquad (2.25)$$

which is a quadratic form with rank = k and the matrix under the [ ] in (2.25) is idempotent and symmetric.

$$Z_1 = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}, \text{ where } z_i \sim N(0, \sigma^2)$$

Hence, Q is distributed as $\chi^2 \sigma^2$ with k d.f. Therefore, to test $H_0 : b_i' \Theta = 0;\ i = 1, 2, ..., k$ we may employ F-test given by

$$F = \frac{Y' A_1 (A_1' A_1)^{-1} B_1' [B_1 (A_1' A_1)^{-1} B_1']^{-1} B_1 (A_1' A_1)^{-1} A_1' Y / k}{Y'Y - \hat{\Theta}' A' Y / (n - r)}$$

with k and $(n - r)$ degrees of freedom.

Example 2.3 : If $E(Y) = A\Theta$, a non-negatively definite matrix and $D(Y) = V$ show that

$$\hat{\Theta} = (A' V^{-1} A)^{-1} A' V^{-1} Y$$

provided $(A' V^{-1} A)^{-1}$ exists. Hence, obtain the BLUE of $b' \Theta$.

3

THEORY OF LINEAR ESTIMATION BASED ON G-INVERSE

3.1 CONCEPT OF A GENERALISED INVERSE

If $(A)_{m \times n}$ with $\rho(A) = r \leq \min(m, n)$, then A^{-1} is not defined unless $m = n$ and $r = n$ (or m), i.e. unless A is a non-singular matrix. Accordingly, a generalised inverse $(A^g)_{n \times m}$ (or A^-) is defined as

$$AA^g A = 1$$

It can be shown that A^g always exists but is not unique in general (Biswas S, 1993). The simplest form of generalised inverse (g-inverse) of any matrix A is to partition A as

$$(A)_{m \times n} = \begin{bmatrix} (A_1)_{r \times r} & (A_2)_{r \times (n-r)} \\ (A_3)_{m-r \times r} & (A_4)_{(m-r) \times (n-r)} \end{bmatrix}_{m \times n}$$

If $\rho(A) = r$ and if we assume without any loss of generality $\rho(A_1) = r$ then A_1 being an r-square non-singular matrix A_1^{-1} exists. Under this situation one form of g-inverse is

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$$A^g = \begin{bmatrix} (A_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

where 0's stands for null matrices of appropriate order. One can verify in this case that $AA^gA = A$. In fact any A^g of the form

$$A^g = Q \begin{bmatrix} I_r & U \\ V & W \end{bmatrix} P$$

for arbitrary U, V and W is a generalised inverse of A, where

$$A = P^{-1} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$$

where P and Q are two non-singular matrices. This shows that any number of generalised inverses can be constructed corresponding to any matrix A with any rank and order.

3.1.1 Special Forms of *g*-inverses

(*a*) *Reflexive generalised inverse :*

If $(A^r)_{n \times m}$ is a matrix which satisfies the following conditions :

(*i*) $AA^rA = A$

and (*ii*) $A^rAA^r = A^r$

then A^r is called the ***reflexive g-inverse*** of A.

(*b*) *Left Weak g-inverse :*

If $(A^\omega)_{n \times m}$ is a matrix satisfying

(*i*) $AA^\omega A = A$

(*ii*) $A^\omega AA^\omega = A$

and (*iii*) $(A^\omega A)' = A^\omega A$

then A^ω is called a ***left weak g-inverse*** of A.

(*c*) *Right Weak g-inverse :*

If A^n is a matrix satisfying

(*i*) $AA^nA = A$

(*ii*) $A^nAA^n = A^n$

and (*iii*) $(A^nA)' = A^nA$

then A^n is a ***right weak g-inverse.***

(*d*) *Moore and Penrose (M.P) g-inverse:*

If $(A^+)_{n \times m}$ satisfies the following:

(*i*) $AA^+A = A$

(*ii*) $A^+AA^+ = A^+$

(*iii*) $(A^+A)' = A^+A$

and (*iv*) $(AA^+)' = AA^+$

then A^+ is called ***M.P. g-inverse***

It can be shown that A^+ is unique.

3.1.2 Construction of Different Forms of *g*-inverses

A^r (reflexive g-inverse) can be constructed as

$$A^r = A^{g_1}AA^{g_2} \tag{3.1}$$

where A^{g_1} and A^{g_2} are any two g-inverses of A and A^ω may be constructed as :

$$A^\omega = A'(A'A)^r \tag{3.2}$$

where $(A'A)^r$ is a reflexive g-inverse of $A'A$ constructed as in (3.1). Similarly, A^n may be constructed as

$$A^n = (A'A)^rA' \tag{3.3}$$

Finally A^+ may be constructed as

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$$A^+ = A^{\omega} A A^{n} \tag{3.4}$$

where A^n and A^{ω} are right and left weak g-inverses of A.

3.1.3 Some Important Properties of A^+

(i) $(A^+)^+ = A$

(ii) $(A')^+ = (A^+)'$

(iii) $(AA')^+ = (A^+)' A^+$ and $(A'A)^+ = A^+ (A')^+$

(iv) $A^+ = (A'A)^{\omega} A'$ for any choice of $(A'A)^{\omega}$

$= A (AA')^{n}$ for any choice of AA'

(v) AA^+, A^+A, $I - A^+A$ and $I - AA^+$ are symmetric and idempotent matrices.

(vi) $(AB)^+ = B^+ A^+$

provided B is full column rank and A is full row rank.

3.2 Alternative Proof of Markov's Theorem

If $E(Y)_{n\times 1} = (A)_{n\times m} (\Theta)_{m\times 1}$

and $D(Y) = \sigma^2 I$, then

(i) the BLUE of a linear parametric function $b'\Theta$ is given by $b'\hat{\Theta}$ where $\hat{\Theta}$ satisfies the equation given by

$$A'Y = A'A\hat{\Theta}$$

(ii) an estimate of σ^2 is given by

$$\hat{\sigma}^2 = \frac{(Y - A\hat{\Theta})'(Y - A\hat{\Theta})}{(n - r)}$$

where $r = \rho(A)$

Proof (i) : If $b'\Theta$ is estimable then $\exists$ a linear function $C'Y$ ∋

$$E(C'Y) = b'\Theta \text{ or } C'A\Theta = b'\Theta, \forall \Theta$$

$$\Rightarrow \quad C'A = b' \Rightarrow A'C = b$$

Now consider $b'A^+Y$

as A^+Y is the L.S. solution of $Y = A\Theta + U$

We have

$$E(b'A^+Y) = b'A^+E(Y)$$

$$= b'A^+A\Theta$$

Again $b' = C'A$

$$\Rightarrow \quad E(b'A^+Y) = C'AA^+A\Theta$$

$$= C'A\Theta = b'\Theta$$

Next, we shall show that $b'A^+Y$ is BLUE of $B'\Theta$ and it has the property of uniqueness.

Let us consider the normal equation

$$A'Y = A'A\Theta$$

The particular solution is given by

$$\tilde{\Theta} = (A'A)^+ A'Y = A^+Y \tag{3.5}$$

and the general solution is given by

$$\hat{\Theta} = (A'A^+)A'Y + [I_m - (A'A)^+(A'A)]Z$$

where Z is an arbitrary column vector

$$\Rightarrow \quad b'\hat{\Theta} = b'(A'A^+)A'Y + b'[I_m - (A'A)^+(A'A)]Z \tag{3.6}$$

Using $b'\tilde{\Theta} = b'A^+Y + b'[I_m - (A^+A)]Z$

$$= b'A^+Y + b'Z - b'(A^+A)Z$$

Putting $b' = C'A$ (condition of unbiasedness)

$$b'\hat{\Theta} = b'A^+Y + b'Z - C'A(A^+A)Z$$

$$= b'A^+Y + C'AZ - C'AZ = b'A^+Y \tag{3.7}$$

$$\text{Var}(b'\hat{\Theta}) = \text{Var}(b'A^+Y)$$

$$= b'A^+D(Y)(A^+)'b$$

$$= b'A^+\sigma^2 I(A^+)'b$$

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$$= \sigma^2 [b' A^+ (A')^+ b]$$

$$= \sigma^2 [b' (A'A)^+ b] \tag{3.8}$$

Finally, we shall show that $b' \hat{\Theta}$ is unique and has the minimum variance in the class of all unbiased estimators. If possible, let $d'Y$ be another unbiased estimator of $b'\Theta$ which has a variance less than $b'A^+Y$.

Now

$$\text{Var}(d'Y) = d'D(Y)\,d$$

$$= d'\sigma^2 I d$$

$$= \sigma^2 d'd \tag{3.9}$$

$$\text{Var}(b' \hat{\Theta}) = \text{Var}(b'A + Y)$$

$$= b' A^+ D(Y) (A^+)' b$$

$$= b' A^+ \sigma^2 I (A^+)' b$$

$$= \sigma^2 b' (A'A)^+ b \tag{3.10}$$

$$(3.9) - (3.10) \Rightarrow \sigma^2 [d'd - b' (A'A)^+ b]$$

$$= \sigma^2 (d - (A^+)' b)' \; (d - (A^+)' b) \geq 0 \tag{3.11}$$

Since $d'A\Theta = b'\Theta \Rightarrow d'A = b' \Rightarrow d' = b' A^+$

R.H.S. of (3.11)

$$= \sigma^2 [d'd - d' (A^+)' b - b' (A^+) d + b' (A^+) (A^+)' b]$$

$$= \sigma^2 [d'd - b' (A^+) (A^+)' b - b' (A^+) (A^+)' b + b' (A^+) (A^+)' b]$$

$$= \sigma^2 [d'd - b' (A'A)^+ b] = \text{L.H.S.}$$

$$= \text{Var}(d'Y) - \text{Var}(b' \hat{\Theta}) \geq 0, \text{ it follows}$$

$$\text{Var}(b' \hat{\Theta}) \leq \text{Var}(d'Y), \text{ for all choice of } d \tag{3.12}$$

Thus, $b' \hat{\Theta}$ is the most efficient in the class of all unbiased linear estimate functions.

Equality on the R.H.S. of (3.11) holds when

$$d = (A^+)' b$$

i.e. $$d'd = b' A^+ (A^+)' b = b' (A'A)^+ b$$

Finally, $b' \hat{\Theta} = b' A^+ Y$ is unique as A^+ is unique. (3.13)

Proof (*ii*) : To show that

$$\sigma^2 = \frac{(Y - A \hat{\Theta})' \; (Y - A \hat{\Theta})}{(n - r)} \tag{3.14}$$

where $A'Y = A'A \hat{\Theta}$ are the normal equations

We have,

$$(Y - A \hat{\Theta}) = (I - AA^+) \; (Y - A\Theta) \tag{3.15}$$

[as R.H.S. of (3.15)$= Y - A\Theta - AA^+Y + AA^+A\Theta$

$$= Y - A\Theta - AA^+Y + A\Theta$$

$$= Y - AA^+Y = Y - A \hat{\Theta} = \text{L.H.S.}]$$

Note : $(I - AA^+)$ is symmetric and idempotent matrix

$$\Rightarrow \quad tr(I - AA^+) = \rho(I - AA^+) = n - r$$

Now consider (3.16)

$$E[(Y - A \hat{\Theta})' \; (Y - A \hat{\Theta})]$$

$$= E[(Y - A\Theta)' (I - AA^+)^2 (Y - A\Theta)], \text{ by (3.15)}$$

$$= E[(Y - A\Theta)' \; (I - AA^+) \; (Y - A\Theta)] \tag{3.17}$$

[as $(I - AA^+)$ is an idempotent matrix]

$$= \sigma^2 [tr(I - AA^+)] + E(Y - A\Theta)' (I - AA^+) E(Y - A\Theta)$$

[as $E(Y'GY) = \sigma^2 (tr\, G) + E(Y')\, GE(Y)$, by (1.74)]

$$= \sigma^2 (n - r)$$

[since $E(Y) = A\Theta$ and $tr(I - AA^+) = n - r$, by (3.16)]

Thus an unbiased estimate of σ^2 is given by (3.14).

This proves Gauss–Markov theorem rigourously by unified linear model theory approach.

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3.3 TESTING OF LINEAR HYPOTHESIS BY USING M.P.G.-INVERSE (Unified Linear Model Approach)

Here in general, our problem is to minimise

$$(Y - A\Theta)' (Y - A\Theta)$$

under a set of linear constraints of the parameters say, $(B)_{n\times m}(\Theta)_{m\times 1} = 0$, where B is any $n \times m$ matrix. We unconditionally minimise

$$Q = (Y - A\Theta)'(Y - A\Theta) + 2\lambda' B\Theta$$

where the column vector $(\lambda')_{n\times 1}$ is the Langrangian Multiplier

$$Q = Y'Y - \Theta' A'Y - Y'A\Theta + \Theta' A'A\Theta + 2\lambda' B\Theta$$
$$= Y'Y - 2\Theta' A'Y + \Theta' A'A\Theta + 2\lambda' B\Theta$$

$$\partial Q/\partial\Theta = 0 \Rightarrow -2A'Y + 2A'A\Theta + 2B'\lambda = 0$$

In other words, we have,

$$A'A\Theta = A'Y - B'\lambda \tag{3.18}$$

Now for any $(B)_{n\times m}$ with $\rho(B) = t$ say, we can have

$$(A)'_{m\times n}(A)_{n\times m}(C)_{m\times n} = (B')_{m\times n} \tag{3.19}$$

(3.19) when substituted in (3.18) gives

$$A'A\Theta = A'Y - A'AC\lambda \tag{3.20}$$

Premultiplying both sides of (3.20) by C' we have

$$C'A'A\Theta = C'A'Y - C'A'AC\lambda$$

$$\Rightarrow \quad C'A'(Y - A\Theta) = C'A'AC\lambda$$

Now
$$\lambda = (C'A'AC)^{+} C'A'(Y - A\Theta) \tag{3.21}$$

Again from (3.18) we have

$$A'(Y - A\Theta) = B'\lambda$$

$$= B'(C'A'AC)^{+}[C'A'(Y - A\Theta)] \tag{3.22}$$

or
$$[A' - B'(C'A'AC)^{+} C'A'](Y - A\Theta) = 0 \tag{3.23}$$

Now R.S.S $= (Y - A\hat{\Theta})'(Y - A\hat{\Theta})$

$$= (Y - A\Theta)'(I - AA^{+})^2 (Y - A\Theta)$$
$$= (Y - A\Theta)'(I - AA^{+})(Y - A\Theta)$$

$(I - AA^{+})$ being idempotent

Again, $(Y - A\Theta)'(Y - A\Theta)$

$$= (Y - A\Theta)'(I - AA^{+})(Y - A\Theta) + (Y - A\Theta)'(A^{+})$$
$$\times [A' - B'(C'A'AC)^{+} C'A'](Y - A\Theta) + (Y - A\Theta)'$$
$$\times (A^{+})'[B'(C'A'AC)^{+} C'A'](Y - A\Theta) \tag{3.24}$$

Denote 1st term of (3.24) by I

2nd term of (3.24) by II

and 3rd term of (3.24) by III

Note that $II + III = (Y - A\Theta)'(A^{+})'A'(Y - A\Theta)$

$$= (Y - A\Theta)'(AA^{+})'(Y - A\Theta)$$
$$= (Y - A\Theta)'(AA^{+})(Y - A\Theta)$$

and $II + III + I$ $= (Y - A\Theta)'(Y - A\Theta)$

$$= \text{L.H.S. of (3.24)}$$

Now $(Y - A\Theta)'(I - AA^{+})(Y - A\Theta)$ is a quadratic form of the type $Z'BZ$, where $(Y - A\Theta) = Z$ and $(I - AA^{+}) = B$, an idempotent matrix which is distributed as $\chi^2\sigma^2$ with

$$\text{d.f} = \rho(I - AA^{+}) = n - r$$

[for proof see Biswas, *S* (1993)]

By similar reasoning $II \sim \chi^2\sigma^2$ with $(r - t)$ degrees of freedom, where $\rho(A) = r$, $\rho(B) = t$ and has been reduced to zero by putting (3.23) in (3.24) and $III \sim \chi^2\sigma^2$ with $\rho(B) = t$ degrees of freedom.

Since $\rho(B) = \rho(B') = \rho(A'AC) \le \rho(C'A'AC)$

$$= \rho[BA)^{+}(BA^{+})']$$

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$$= \rho\,[B\,(A'A)^+\,B'] \le \rho\,(B)$$

i.e. $\rho\,(B) \le \rho\,[B\,(A'A)^+\,B']$

$\Rightarrow$ $\rho\,(B) = \rho\,[B\,(A'A)^+\,B']$

Now consider

$$III = (Y - A\,\Theta)'\,(A^+)'\,[B'\,(C'\,A'AC)^+\,C'\,A']\,(Y - A\,\Theta)$$

We have $A'AC = B'$

$\Rightarrow$ $C'\,A'A = B$

$\Rightarrow$ $C'\,A' = BA^+$

$\Rightarrow$ $AC = (BA^+)'$

$\therefore$ $C'A'AC = (BA^+)\,(BA^+)'$

Putting the same in the R.H.S of *III* we get

$$III = (Y - A\,\Theta)'\,(A^+)'\,[B'\,\{(BA^+)\,(BA^+)'\}\,(Y - A\,\Theta)\,C'A']$$

$$= (Y - A\,\Theta)'\,(A^+)'\,[B'\,\{BA^+(A')^+\,B'\}^+\,C'A']\,(Y - A\,\Theta)$$

$$= (Y - A\,\Theta)'\,(A^+)'\,[B'\,\{B\,(A'A)^+\,B'\}^+\,BA^+]\,(Y - A\,\Theta)$$

$$= (Y - A\,\Theta)'\,(A^+)'\,[B'\,\{B\,(A'A)^+\,B'\}^+\,(BA^+\,Y - BA^+\,A\,\Theta)]$$

$$= (BA^+Y - BA^+\,A\,\Theta)'\,[[B\,(A'A)^+\,B'\}^+\,(BA^+Y - BA^+\,A\Theta)]$$

$$= (BA^+Y - B\,\Theta)'\,[[B\,(A'A)^+\,B'\}^+\,(BA^+\,Y - BA^+\,A\,\Theta)]$$

Under $H_0 : B\,\Theta = 0$

$\Rightarrow (BA^+Y)'\,[[B\,(A'A)^+\,B'\}^+\,BA^+\,Y] \sim \chi^2\,\sigma^2$ with t d.f.

Hence, $$F = \frac{(BA^+\,Y)'\,[[\{B\,(A'A)^+\,B'\}^+\,BA^+Y]/t}{Y'Y - \hat{\Theta}'\,A'Y/(n-r)} \qquad (3.25)$$

is distributed as Snedecor's F and can be considered as an appropriate statistic to test the linear hypothesis

$B\,\Theta = 0$ against $B\,\Theta \neq 0$

3.3.1 Particular Cases of Testing Simple Null Hypothesis

(*a*) For testing $b'\,\Theta = 0$ against $b'\,\Theta \neq 0$ (i.e. testing a particular linear function).

We have the BLUE of $b'\,\Theta$,

$$b'\,\hat{\Theta} = b'A^+Y$$

and

$$\text{Var}\,(b'\,\hat{\Theta}) = b'A^+\,D\,(Y)\,(A^+)'\,b$$

$$= b'\,A^+\,\sigma^2 I\,(A^+)'\,b$$

$$= \sigma^2\,b'A^+\,(A^+)'\,b$$

$\therefore$ $$t = \frac{\{b'\,A^+Y - 0\}/\sqrt{b'A^+\,(A^+)'\,b\}}}{Y'Y - \hat{\Theta}'\,A'Y/(n-r)} \qquad (3.26)$$

is distributed as Student's t being the ratio of a Normal and χ^2 variate with $(n-r)$ d.f. This can be used for testing $b'\,\Theta = 0$ against $b'\,\Theta \neq 0$.

(*b*) For testing $\Theta = 0$ against $\Theta \neq 0$

Obviously, the above test is trivial but can be considered independently. We have,

$$\text{R.S.S (unrestricted)} = (Y - A\,\hat{\Theta})'\,(Y - A\,\hat{\Theta})$$

$$= (Y - A\,\Theta)'\,(I - AA^+)\,(Y - A\,\Theta)$$

$$= (Y' - \Theta'\,A')\,(I - AA^+)\,(Y - A\,\Theta)$$

$$= Y'\,(I - AA^+)\,Y, \text{ under } \Theta = 0$$

$$= Y'Y - Y'\,AA^+Y$$

$$= Y'Y - Y'A\,\hat{\Theta}$$

$$= Y'Y - \hat{\Theta}'\,A'Y$$

$$= \text{Residual S.S under } H_0$$

Total sum of squares $= Y'Y$

R.S.S due to hypothesis $= Y'Y - (Y'Y - \hat{\Theta}'\,A'Y)$, under $\Theta = 0$

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$= \hat{\Theta}' A'Y$ with r d.f.

$\rho(A) = r$ is tested against $Y'Y - \hat{\Theta}' A'Y$ with $(n-r)$ d.f.

$$F = \frac{\hat{\Theta}' A'Y / r}{Y'Y - \hat{\Theta}' A'Y/(n-r)} \qquad (3.27)$$

and

is the appropriate statistic for testing $\Theta = 0$ against $\Theta \neq 0$.

3.4 Other Approaches to the Least Squares Theory (Geometric Solution-Reference: Rao, C.R.)

Let P be a projection matrix which projects every vector α into the estimation space i.e. a vector space spanned by the column vectors of

$$A = (a_1, a_2, \ldots, a_m)$$

Obviously, $PA = A$

It is proved that P is a symmetric and idempotent matrix [Biswas, S (1993)].

Theorem 3.1 : If P is a projection matrix then P is symmetric and idempotent. Conversely all symmetric and idempotent matrices are projection matrices.

Proof : Let P be a projection matrix which projects every vector in V_α (a linear manifold) in itself.

$$PY = Y \text{ if } Y \in V_\alpha$$

If $X \notin V_\alpha \quad PX = Y$

$$\Rightarrow \quad Y = PY = P(PX) = P^2X$$

Also $\quad Y = PX$

$$\Rightarrow \quad (P - P^2)X = 0$$

$$\Rightarrow \quad P = P^2, \text{ i.e. } P \text{ is idempotent}$$

If $X \notin V_\alpha$ and $Y \in V_\alpha$, then

$$X = Y + a \text{ vector } \perp^r V_\alpha$$

$$\Rightarrow \quad (X - Y) = \perp^r V_\alpha$$

$$(X - Y)' PX = 0$$

i.e. $\quad (X - PX)' PX = 0$

$$(X' - X' P') PX = 0$$

$$X'PX - X' P'PX = 0$$

$$X' (P - P'P) X = 0$$

$$\Rightarrow \quad P = P'P$$

$$\Rightarrow \quad P' = P'P$$

i.e. $\quad P = P'$

$\Rightarrow P$ is a symmetric and idempotent matrix.

Converse : If P is symmetric and idempotent, then P is a projection matrix.

Let $Y = PX$ and P is a symmetric and idempotent matrix

$$\Rightarrow \quad P = P^2$$

$$\Rightarrow \quad Y = P^2X = P(PX) = PY$$

$\Rightarrow P$ is a transformation matrix which projects $\forall\ Y \in V_\alpha$ into the same space

Again if $\quad Y = PY = PX$

$$\Rightarrow \quad P(Y - X) = 0$$

$$P'(Y - X) = 0; \text{ since } P = P'$$

i.e. column vector P' is $\perp^r (Y - X)$

$$(Y - X) \perp^r Y = PY$$

$$(X - Y) \perp Y = PY$$

i.e. $X - Y$ is $\perp^r$ any vector in V_α

$$\Rightarrow \quad X = Y + \text{a vector } \perp^r V_\alpha$$

$\Rightarrow Y$ is a projection of X in V_α

$\therefore\ Y = PX \Rightarrow P$ is a projection matrix.

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3.4.1 Construction of BLUE through Projection Matrix

As usual, let $C'Y$ be any linear function, unbiased for $b'\Theta$.

$$E(C'Y) = b'\Theta$$

$$C'A\Theta = b'\Theta \text{ holding } \forall\ \Theta$$

$$C'A = b'$$

Next consider $E(C'PY) = C'PA\Theta$

$$= C'A\Theta \text{ (as } PA = A)$$

$$= b'\Theta$$

$\Rightarrow C'PY$ is unbiased for $b'\Theta$

Also $\text{Var}(C'Y) = \text{Var}[C'(I-P)Y + C'PY]$

$$= \text{Var}[C'(I-P)Y] + \text{Var}(C'PY)$$

$$+ 2\,\text{Cov}[C'(I-P)Y, C'PY]$$

$\therefore$ $\text{Var}(C'Y) = \text{Var}[C'(I-P)Y] + \text{Var}(C'PY)$

$\text{Cov}[C'(I-P)Y, C'PY]$

$$= C'(I-P)E(YY')P'C - C'(I-P)E(Y)E(Y')P'C = \sigma^2$$

$$= \sigma^2\{C'(I-P)P'C\};\ \text{as } E(YY') - E(Y)E(Y') = \sigma^2$$

$$= \sigma^2\{C'P'C - C'PP'C\} = 0$$

[since $P' = P$ and $P^2 = P$]

Hence, $V(C'Y) \geq \text{Var}(C'PY)$

3.4.2 Derivation of Normal Equations by Projections

We consider the projection of Y in the estimation space $V_\alpha(\alpha_1, \alpha_2, \ldots, \alpha_m)$

Now Y = Projection of Y on V_α + Projection of Y on $\perp^r V_\alpha$

$$= PY + \text{Projection vector } \perp^r V_\alpha$$

$\Rightarrow$ $Y - PY \perp^r V_\alpha$

whereas $Y - PY \in V_\alpha$ (3.28)

We can write (3.29)

$$PY = \alpha_1\hat{\Theta}_1 + \alpha_2\hat{\Theta}_2 + \ldots + \alpha_m\hat{\Theta}_m$$

$$= (\alpha_1, \alpha_2 \ldots, \alpha_m)(\hat{\Theta}_1, \hat{\Theta}_2 \ldots, \hat{\Theta}_m)'$$

$$= A\hat{\Theta}$$

using $PY = A\hat{\Theta}$, $Y - A\hat{\Theta}$ is a vector $\perp^r V_\alpha$ (3.30)

i.e. $A'(Y - A\hat{\Theta}) = 0$

$$A'Y = A'A\hat{\Theta}$$

$$\hat{\Theta} = (A'A)^{-1}A'Y \text{ if } (A'A)^{-1} \text{ exists}$$

$$= A^+Y \text{ otherwise}$$

Now to obtain P

$$A\hat{\Theta} = A(A'A)^{-1}A'Y$$

or $$= AA^+Y$$

$$PY = A\hat{\Theta} = A(A'A)^{-1}A'Y$$

or $$= AA^+Y$$

Theorem 3.2 : $C'PY$ has the minimum variance in the class of all unbiased estimators (BLUE).

Proof : Let $d'Y$ be unbiased for $b'\Theta$

$$E(d'Y) = b'\Theta$$

$$d'A\Theta = b'\Theta \text{ holding } \forall\ \Theta$$

$\Rightarrow$ $$d'A = b'$$

Also $$C'A = b'$$

$\Rightarrow$ $$(C' - d')A = 0$$

where $A = [\alpha_1, \alpha_2 \ldots, \alpha_m]$ is the vector space spanning estimation space.

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$\therefore (C' - d')$ must be $\perp' A$

i.e. $A'(C - d) = 0$

Then $P(C - d) = 0$

$\Rightarrow$ $PC = Pd$

Transposing $C'PY = d'PY$ (as $P^2 = P$)

$\text{Var}(C'PY) = \text{Var}(d'PY)$

Again $\text{Var}(d'Y) \geq \text{Var}(d'PY)$

$= \text{Var}(C'PY)$

$\text{Var}(C'PY) \leq \text{Var}(d'Y)$

$\Rightarrow C'PY$ is BLUE for $b'\Theta$

where $P = A(A'A)^{-1}$ if $(A'A)^{-1}$ exists

$= AA^{+}$ if $A'A$ is singular

4

MISCELLANEOUS PROBLEMS IN LEAST SQUARES THEORY

4.1 Simultaneous Estimation of Parametric Functions

Consider the general set up

$$(Y_2 \times \Theta, \Sigma)$$

Let $b_1'\hat{\Theta},\ b_2'\hat{\Theta}, \ldots, b_k'\hat{\Theta}$ (4.1)

be the individual O.L.S.E. of

$b_1'\Theta,\ b_2'\Theta, \ldots,\ b_k'\Theta$, respectively (4.2)

Also let A be the dispersion matrix of (4.1). We have the following optimal properties of OLS Estimation given from (i) to (iv)

Let $C_1'Y,\ C_2'Y \ldots C_k'Y$ (4.3)

be unbiased estimators of

$b_1'\Theta,\ b_2'\Theta \ldots b_k'\Theta$ respectively

and let B be their dispersion matrix. Then $(B - A)$ is non-negatively definite.

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$\Rightarrow$ (i) Trace $(B) \geq$ Trace (A)

(ii) $|B| \geq |A|$

(iii) Tr $(QB) \geq$ Tr (QA) $\forall$ n.n.d Q (non-negatively definite quadratic form).

(iv) max. latent root of $B \geq$ max. latent root of A

To prove the same, we consider

$$a_1 b_1' \Theta + a_2 b_2' \Theta + \ldots + a_k b_k' \Theta \tag{4.4}$$

whose least squares estimator is

$$a_1 b_1' \hat{\Theta} + a_2 b_2' \hat{\Theta} + \ldots + a_k b_k' \hat{\Theta} \tag{4.5}$$

with variance $a'Aa$

where $a' = (a_1, \ldots, a_k)$

The variance of the alternative estimate

$$a_1 C_1'Y + a_2 C_2'Y + \ldots + a_k C_k'Y = a'Ba$$

Then we have

$$a'Ba \geq a'Aa$$

which proves all the properties (i) through (iv).

4.2 Some Special Topics on Linear Regression

4.2.1 Least Squares Theory when Parameters are Random Variables

Consider

$$(Y)_{n\times 1} = (A)_{n\times m}(\Theta)_{n\times 1} + (U)_{n\times 1}$$

where Θ is a r.v. with

$$E(\Theta) = \mu \text{ and } D(\Theta) = T$$

As usual

$$E(Y \mid \Theta) = A\Theta \text{ and } D(Y \mid \Theta) = \Sigma \tag{4.6}$$

$$D(Y) = E[D(Y \mid \Theta)] + D[E(Y \mid \Theta)]$$

$$= E[\Sigma] + D[A\Theta]$$
$$= \Sigma + AD(\Theta)A'$$
$$= \Sigma + ATA'$$

Let $b'\Theta$ is a parametric function. When Θ is a r.v. $b'\Theta$ is also a r.v. (4.7)

Consider

$$\text{Cov}(Y, b'\Theta) = E[\text{Cov}(Y, b'\Theta) \mid \Theta] + \text{Cov}[E(Y \mid \Theta), E(b'\Theta \mid \Theta)]$$
$$= AD(\Theta)b; \quad [\text{as } E \text{ Cov}(Y, b'\Theta) \mid \Theta = 0]$$
$$= ATb$$

The problem is reduced to minimising Var $(b'\Theta - C'Y)$ subject to $E(b'\Theta - C'Y) = 0$, i.e. subject to (4.8)

$$b'\mu = C'A\mu$$

i.e. $$b' = C'A$$

We obtain a linear function $C'Y$ ∋

$$E(b'\Theta - C'Y) = 0$$

subject to Var $(b'\Theta - C'Y)$ is a minimum (4.9)

i.e. Min Var $(b'\Theta - C'Y)$

subject to $b'\mu = C'A\mu$

or $b' = C'A$

or $b = A'C$

Now putting (4.6), (4.7) and (4.8) in (4.9)

Min Var $(b'\Theta - C'Y)$

Also $Q = \text{Var}(b'\Theta - C'Y) = \text{Var}(b'\Theta) + \text{Var}(C'Y) - 2\,\text{Cov}(b'\Theta, C'Y)$

$$= b'D(\Theta)b + C'D(Y)C - 2C'ATb$$
$$= b'Tb + C'(\Sigma + ATA')C - 2C'ATb$$

subject to $b' = C'A$ (4.10)

or $Q = b'Tb + C'(\Sigma + ATA')C - 2C'ATb$

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$$b = A'C \tag{4.11}$$

subject to

Setting $\partial Q/\partial C = 0$

$2C(\Sigma + ATA') - 2ATb = 0$

We have, $C(\Sigma + ATA') = ATb$

$$C = (\Sigma + ATA')^{-1} ATb \tag{4.12}$$

$\Rightarrow$ Using the relation $b = A'C$ we have the optimum C given by C^* which will satisfy (4.12)

i.e. $C^* = (\Sigma + ATA')^{-1} ATb,$

$$= (\Sigma + ATA')^{-1} ATA'C \tag{4.13}$$

Also using

$$(\Sigma + ATA')^{-1} = [\Sigma^{-1} - \Sigma^{-1}A(A'\Sigma^{-1}A + T^{-1})A'\Sigma^{-1}$$

[Rao, C.R. (1983); Biswas, S. (1993)]

it can be shown that

$$C^* = \Sigma^{-1}A(T^{-1} + A'\Sigma^{-1}A)^{-1}$$

$$= (\Sigma + ATA')^{-1} ATb$$

The prediction variance is

$$\text{Var}[b'\Theta - (C^*)'Y] = A'TA - A'TA'C^*$$

$$= A'(T^{-1} + A'\Sigma^{-1}A)^{-1} b$$

Proof consists of varifying that the choice of C^* satisfies

$E(b'\Theta - C'Y) = 0$ and $\text{Var}(b'\Theta - C'Y)$ is minimum

i.e. for any other choice of C^*, say $C^@$

$$\text{Var}(b'\Theta - C^{@\prime}Y) \geq \text{Var}(b'\Theta - C^{*\prime}Y).$$

5

REGRESSION THEORY AND ITS APPLICATION

5.1 Concept of a Regression Function

Regression of Y on $X_1\, X_2 \ldots X_p$ is defined as

$$E(Y \mid X_1,\ X_2,\ ...,\ X_p) \tag{5.1}$$

where Y is the ***Criterion or Dependent variable*** and X_i's are ***predictor variables.***

Similarly regression of X on Y is defined as

$$E(X \mid Y_1,\ Y_2,\ ...,\ Y_p) \tag{5.2}$$

In defining regression of Y on X, $(X_1, X_2, ..., X_p)'$ are considered as non stochastic as they are given. Similarly for X on Y, $(Y_1, Y_2, ..., Y_p)'$ are taken as non stochstic. Estimation of regression line of Y on $X = x$ given by the regression equation

$$Y = \beta_0 + \beta_1 x \tag{5.3}$$

is same as fitting a straight line on the data of Y for a given value of the r.v., say $X = x$. However, the estimation of the regression parameters are not the same as that of a curve fitting when regression is non-linear.

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5.2 Estimation and Standard Errors of the Regression Parameters

We recall the linear model $Y = A\Theta + U$

Replacing Θ by β and A by X, we rewrite the same as

$$Y = X\beta + U \tag{5.4}$$

which can be written in matrix notation as follows:

$$\begin{bmatrix} y_1 \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_n \end{bmatrix}$$

We have the O.L.S. of β given by $\hat{\beta}$

$$\hat{\beta} = (X'X)^{-1} X'Y \tag{5.5}$$

$$D(\hat{\beta}) = (X'X)^{-1} \sigma^2 \tag{5.6}$$

If $$D(U) = \sigma^2 I \tag{5.7}$$

$$\hat{\beta} = \left[\begin{pmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ x_1 & x_2 & \dots & x_n \end{pmatrix} \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \right]^{-1}$$

$$\times \begin{pmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ x_1 & x_2 & \dots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$= \begin{pmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{pmatrix}$$

$$= \begin{bmatrix} \frac{\sum x_i^2}{n \sum (x_i - \bar{x})^2} & \frac{-\bar{x}}{n \sum (x_i - \bar{x})^2} \\ \frac{-\bar{x}}{\sum (x_i - \bar{x})^2} & \frac{1}{\sum (x_i - \bar{x})^2} \end{bmatrix} \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{bmatrix}$$

$$\begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} \frac{\sum x_i^2}{n \sum (x_i - \bar{x})^2} & \frac{-\bar{x} \sum x_i y_i}{\sum (x_i - \bar{x})^2} \\ \frac{-\bar{x} \sum y_i}{\sum (x_i - \bar{x})^2} & \frac{\sum x_i y_i}{\sum (x_i - \bar{x})^2} \end{bmatrix}$$

$$\hat{\beta}_1 = \frac{\sum x_i y_i - n \bar{x} \bar{y}}{\sum (x_i - \bar{x})^2} \tag{5.8}$$

As usual

$$\hat{\beta}_0 = \frac{\sum x_i^2 - n \bar{x} \sum x_i y_i}{n \sum (x_i - \bar{x})^2} \tag{5.9}$$

Standard error of β_0 and β_1 are obtainable from

$$D(\hat{\beta}) = \begin{bmatrix} \frac{\sum x_i^2}{n \sum (x_i - \bar{x})^2} & \frac{-\bar{x}}{\sum (x_i - \bar{x})^2} \\ \frac{-\bar{x}}{\sum (x_i - \bar{x})^2} & \frac{1}{\sum (x_i - \bar{x})^2} \end{bmatrix} \sigma^2$$

$$= \begin{bmatrix} \frac{\sigma^2 \sum x_i^2}{n \sum (x_i - \bar{x})^2} & \frac{-\bar{x} \sigma^2}{\sum (x_i - \bar{x})^2} \\ \frac{-\bar{x} \sigma^2}{\sum (x_i - \bar{x})^2} & \frac{\sigma^2}{\sum (x_i - \bar{x})^2} \end{bmatrix}$$

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$$\text{Var}(\hat{\beta}_0) = \frac{\sigma^2 \Sigma x_i^2}{n \Sigma (x_i - \bar{x})^2} \tag{5.10}$$

$$Cov(\hat{\beta}_0, \hat{\beta}_1) = \frac{-\bar{x}\sigma^2}{\Sigma (x_i - \bar{x})^2} \tag{5.11}$$

$$\text{Var}(\hat{\beta}_1) = \frac{\sigma^2}{\Sigma (x_i - \bar{x})^2} \tag{5.12}$$

$$S.E(\hat{\beta}_0) = \sigma \sqrt{\frac{\Sigma x_i^2}{n \Sigma (x_i - \bar{x})^2}} \tag{5.13}$$

$$S.E(\hat{\beta}_1) = \sigma \sqrt{\frac{1}{\Sigma (x_i - \bar{x})^2}} \tag{5.14}$$

If the value of σ is not known a priori then its estimate

$$\hat{\sigma}^2 = \frac{\Sigma (x_i - \bar{x})^2}{n-1} \tag{5.15}$$

may be employed.

S.S due to departure from regression $= \Sigma (y_i - \hat{y}_i)^2$

$$= \Sigma (y_i - \bar{y} + \bar{y} - \hat{y}_i)^2$$

$$= \Sigma (y_i - \bar{y})^2 + \Sigma (\hat{y}_i - \bar{y})^2 - 2 \Sigma (y_i - \bar{y})(\hat{y}_i - \bar{y})$$

$$= \Sigma (y_i - \bar{y})^2 + \Sigma (\hat{y}_i - \bar{y})^2 - 2 \Sigma (y_i - \bar{y}) b_1 (x_i - \bar{x})$$

$$[\text{since } (\hat{y}_i - \bar{y}) = b_1 (x_i - \bar{x})]$$

$$= \Sigma (y_i - \bar{y})^2 + \Sigma (\hat{y}_i - \bar{y})^2 - 2 b_1^2 \Sigma (x_i - \bar{x})^2$$

$$[\text{since } \Sigma (y_i - \bar{y})(x_i - \bar{x}) = b_1 \Sigma (x_i - \bar{x})^2$$

$$= \Sigma (y_i - \bar{y})^2 + \Sigma (\hat{y}_i - \bar{y})^2 - 2 \Sigma (\hat{y}_i - \bar{y})^2$$

$$= \Sigma (y_i - \bar{y})^2 - \Sigma (\hat{y}_i - \bar{y})^2 \tag{5.16}$$

$$\Rightarrow \quad \Sigma (y_i - \hat{y}_i)^2 + \Sigma (\hat{y}_i - \bar{y})^2 = \Sigma (y_i - \bar{y})^2$$

Now, $\Sigma (y_i - \bar{y})^2$ = S.S from the mean = T.S.S. (5.17)

$$= \Sigma y_i^2 - \frac{(\Sigma y_i)^2}{n}$$

Also, $\Sigma (\hat{y}_i - \bar{y})^2$ = S.S. due to regression (5.18)

$$= b_1^2 \Sigma (x_i - \bar{x})^2$$

$$= b_1 [b_1 \Sigma (x_i - \bar{x})^2]$$

$$= b_1 \Sigma (x_i - \bar{x})(y_i - \bar{y})$$

$$= b_1 \left[\Sigma x_i y_i - \frac{\Sigma x_i \Sigma y_i}{n} \right] \tag{5.19}$$

$\therefore$ $\Sigma (y_i - \hat{y}_i)^2$ = S.S. due to departure from regression

$$= \Sigma y_i^2 - \frac{(\Sigma y_i)^2}{n} - b_1 \left[\Sigma x_i y_i - \frac{(\Sigma x_i)(\Sigma y_i)}{n} \right] \tag{5.20}$$

5.3 ANOVA TABLE FOR THE TEST OF SIGNIFICANCE OF REGRESSION COEFFICIENTS

Sources of Variation	Sum of Squares	d.f.	Mean Squares
Regression	$b_1 \left[\Sigma x_i y_i - \frac{\Sigma x_i \Sigma y_i}{n} \right]$	1	$b_1 \left[\Sigma x_i y_i - \frac{\Sigma x_i \Sigma y_i}{n} \right]$
Departure from regression	$\left\{ \Sigma y_i^2 - (\Sigma y_i)^2/n - b_1 \left[\Sigma x_i y_i - \frac{\Sigma x_i \Sigma y_i}{n} \right] \right\}$	$n-2$	$\frac{\text{S.S.}}{n-2}$
Total	$\Sigma y_i^2 - (\Sigma y_i)^2/n$	$n-1$	

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A more general form of ANOVA table is obtainable by incorporating the correction factor for the mean of Y into the table. It is called S.S due to b_0. This is presented below :

Sources of Variation	Sum of Squares	d.f	Mean Squares
Regression (b_0)	$\Sigma y_i = (i)$	1	$(i)/1$
Regression ($b_1 \mid b_0$)	$b_1 [\Sigma x_i y_i - (\Sigma x_i \Sigma y_i)/n] = (ii)$	1	$(ii)/1$
Residual	$[\Sigma y_i^2 - (i) - (ii)] = (iii)$	$n-2$	$(ii)/n-2$
Total (uncorrected for mean)	Σy_i^2	n	

Now we can write

$$SS(b_0) + SS(b_1 \mid b_0) = b_1 \Sigma x_i y_i - n\bar{x}\bar{y} + n\bar{y}^2$$
$$= b_1 \Sigma x_i y_i + n\bar{y}(\bar{y} - b_1 \bar{x})$$
$$= b_1 \Sigma x_i y_i + b_0 \Sigma y_i$$
$$= (b_0,\ b_1) \begin{pmatrix} \Sigma y_i \\ \Sigma x_i y_i \end{pmatrix}$$
$$= b' X' Y$$

$S.S.(b_0) + S.S(b_1 \mid b_0) = S.S$ due to regression with 2 d.f.

Hence, one can write the ANOVA table in matrix form as

Source	S.S	d.f	M.S.
$b = (b_0, b_1)$	$b' X' Y$	2	$b' X' Y/2$
Residual	$Y'Y - b' X' Y'$	$n-2$	$\dfrac{(Y'Y - b' X' Y)}{(n-2)}$
Total (uncorrected)	$Y'Y$	n	

5.4 VARIANCE OF $\hat{Y}$ ($\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X$)

Let X_k be a selected value of X, then

$$\hat{Y}_k = \hat{\beta}_0 + \hat{\beta}_1 X_k$$

Let us define $\tilde{X}_k$ as $\tilde{X}_k' = (1, X_k)$

$$\hat{Y}_k = (1, X_k) \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = \tilde{X}_k' \hat{\beta} \text{ (a scalar)}$$
$$= \hat{\beta}' X_k$$

We can write (5.21)

$$\text{Var}(\hat{Y}_k) = \text{Var}(\hat{\beta}_0) + 2X_k \text{Cov}(\hat{\beta}_0, \hat{\beta}_1) + X_k^2 \text{Var}(\hat{\beta}_1)$$

This quadratic form can be written as

$$\text{Var}(\hat{\beta}_k) = (1, X_k) \begin{bmatrix} \text{Var}(\hat{\beta}_0) & \text{Cov}(\hat{\beta}_0, \hat{\beta}_1) \\ \text{Cov}(\hat{\beta}_0, \hat{\beta}_1) & \text{Var}(\hat{\beta}_0) \end{bmatrix} \begin{bmatrix} 1 \\ X_k \end{bmatrix}$$
$$= X_k' \sigma^2 (X'X)^{-1} X_k = X_k' (X'X)^{-1} X_k \sigma^2 \quad (5.22)$$

Again Multiple Correlation Ceofficient defined as

$$R^2 = \frac{\text{S.S. due to regression}}{\text{Total S.S.}}$$
$$= \frac{\beta' X' Y - n\bar{y}^2}{Y'Y - n\bar{y}^2} \quad (5.23)$$

is a measure of the extent of linear regression.

5.5 OTHER MISCELLANEOUS PROBLEMS IN REGRESSION ANALYSIS

5.5.1 'Extra Sum of Squares' Principle

Often it is a question while ascertaining a suitable regression model, whether a particular term or effect is necessary. It is necessary in the sense that the sum of squares due to regression is increasing because of incorporating the specific term (or the Multiple Correlation Coefficient is increasing on account of that). To be more specific, let us consider the model

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$$Y = \beta_0 + \beta_1 Z_1 + \beta_2 Z_2 + \ldots + \beta_p Z_p \qquad (5.24)$$

where $Z_1, Z_2, \ldots, Z_p$ are known functions of the r.v.'s $X_1, X_2, \ldots, X_p$ respectively; Y being the dependent variable.

Suppose

$$b_0^{(1)},\ b_1^{(1)}, \ldots,\ b_p^{(1)} \qquad (5.25)$$

are the OLSE of $\beta_0, \beta_1, \ldots, \beta_p$ respectively. Let us denote the sum of squares due to regression by

$$SS\,[b_0^{(1)},\ b_1^{(1)},\ \ldots,\ b_p^{(1)}] = S_1 \qquad (5.26)$$

$$\Rightarrow \quad (b_0^{(1)}, b_1^{(1)}, \ldots, b_p^{(1)})\,(x_1\ x_2 \ldots x_p)\begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} = S_1$$

Let the estimate of σ^2 be s^2 based on the residual S.S. of the model. Next let us consider another model by deleting several terms in the earlier model as

$$Y = \beta_0 + \beta_1 Z_1 + \beta_2 Z_2 + \ldots + \beta_q Z_q + \epsilon \qquad (5.27)$$

where $q < p$. Suppose the OLSE of $\beta_0, \beta_1, \ldots, \beta_q$ are

$$b_0^{(2)},\ b_1^{(2)}, \ldots,\ b_q^{(2)} \qquad (5.28)$$

Now it may happen that

$$b_i^{(1)} = b_i^{(2)};\ i = 1, 2, \ldots, q$$

If they are the same; it means that $b_i^{(1)}$ and $b_j^{(1)}$ are orthogonal linear functions for

$$1 \le i \le q;\ q+1 \le j \le p$$

In other words, we have in model (5.24), the first q columns of the X-Matrix are all orthogonal to last $(p-q)$ columns.

Suppose, $\quad S.S\,(b_0^{(2)},\ b_1^{(2)}, \ldots,\ b_q^{(2)}) = S_2 \qquad (5.29)$

Then $S_1 - S_2$ represents the extra sum of squares due to inclusion of the terms $\beta_{q+1} Z_{q+1} + \beta_{q+2} Z_{q+2} + \ldots + \beta_p Z_p$ in model (5.24).

S_1 is distributed with $(p+1)$ d.f.

and S_2 is distributed with $(q+1)$ d.f.

$\therefore$ $(S_1 - S_2)$ is distributed with $(p+1) - (q+1) = (p-q)$ d.f.

Further, assuming that the errors are normally distributed, it follows that

$$S_1 - S_2 \sim \sigma^2 \chi^2_{p-q}$$

independently of s^2 (the estimate of σ^2, error variance). In other words $(S_1 - S_2)/(p-q)$ can be tested against S^2 with ν degrees of freedom. *i.e.* the above ratio is distributed as Snedecor's F with $p-q$, ν degrees of freedom. This leads to the testing of hypothesis

$$H_0 : \beta_{q+1} = \beta_{q+2} = \ldots = \beta_p = 0 \qquad (5.30)$$

Often it is convenient to write $S_1 - S_2$ as

$$S.S\,(b_{q+1} \ldots b_p \mid b_0\, b_1 \ldots b_q) \qquad (5.31)$$

In fact, one defect of this kind of notation is that while the result investigates the differences in the two models, the notations do not display the same. This is read as the sum of squares of $b_{q+1}, \ldots, b_p$ given $b_0, b_1 \ldots, b_q$. However, by continued application of the technique we can obtain

$$SS\,(b_0),\ SS\,(b_1 \mid b_0),\ SS\,(b_2 \mid b_0, b_1) \ldots SS\,(b_p \mid b_0, b_1, \ldots, b_{p-1}).$$

Note that each of the sum of squares are distributed independently of S^2 (estimate of error sum of squares) and can be tested by F-test against S^2 and equal to their mean squares since each of them has one degree of freedom.

Suppose $\quad Z_j = X^j \qquad (5.32)$

then this technique enables to find out the improvement of the model from constant term to a linear function of X, the quadratic functions, the cubic functions of X etc. sequentially.

Similarly in case of polynomial models, the procedure of ***Extra Sum of Squares Principle*** enables us to find out the S.S due to β_0 (constant term model), the S.S due to linear terms, quadratic terms (or second order terms) etc.

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In a nutshell, Extra Sum of Squares principle is actually a special case for testing of general linear hypothesis. Often, in the general case, the extra sum of squares are conventionally calculated from residual sums of squares and not from regression sums of squares.

5.5.2 Effect of Orthogonal Columns in the X-matrix

Suppose, we employ 'Extra Sum of Squares Principles' (ESSP) to obtain

$SS(b_1)$ from the model $Y = \beta_1 X_1 + \epsilon$

$SS(b_1 \mid b_0)$ from the model $Y = \beta_0 + \beta_1 X_1 + \epsilon$

$SS(b_2 \mid b_0, b_1)$ from the model $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \epsilon$

Usually the sum of squares are expected to be different, unless columns of the X-matrix corresponding to b_0, b_1, and b_2 columns are orthogonal. Suppose, we consider a general situation, say

$$Y = Xb + \epsilon \tag{5.33}$$

and X is partitioned into t columns as

$$X = (X_1, X_2, ..., X_t)$$

and a corresponding division of the parametric vector β into t rows as

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_t \end{bmatrix}$$

The model can now be written as

$$E(Y) = X\beta = X_1\beta_1 + X_2\beta_2 + \ldots + X_t\beta_t \tag{5.34}$$

The normal equation is given by

$$X'X\hat{\beta} = X'Y \tag{5.35}$$

where $\hat{\beta}$ is the OLS estimate of β. Now if the columns of X are orthogonal (say, j^{th} column vector is orthogonal to i^{th} column vector $\forall\, j \neq i$) then,

$$X_i' X_j = 0 \;\forall\; i \neq j$$

$$\Rightarrow \quad \text{SS due to } \hat{\beta} = \hat{\beta}'X'Y$$

$$= (\hat{\beta}_1\ \hat{\beta}_2' \ldots \hat{\beta}_t') \begin{pmatrix} X_1'Y \\ X_2'Y \\ \vdots \\ X_t'Y \end{pmatrix}$$

$$= \hat{\beta}_1' X_1'Y + \hat{\beta}_2' X_2'Y + \ldots + \hat{\beta}_t' X_t'Y$$

$$= SS(\hat{\beta}_1) + SS(\hat{\beta}_2) + \ldots + SS(\hat{\beta}_t)$$

In case, for $t = 2$

$$X = (X_1\ X_2);\ X' = \begin{pmatrix} X_1' \\ X_2' \end{pmatrix}$$

The normal equations are

$$X'X\hat{\beta} = X'Y \tag{5.36}$$

$$\Rightarrow \quad \begin{pmatrix} X_1' \\ X_2' \end{pmatrix} (X_1\ X_2) \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{pmatrix} X_1'Y \\ X_2'Y \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{pmatrix} X_1'Y \\ X_2'Y \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} X_1'X_1 & 0 \\ 0 & X_2'X_2 \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{pmatrix} X_1'Y \\ X_2'Y \end{pmatrix}$$

Since all columns of X_1 are $\perp^r$ to all columns of x_2

$$X_1'X_2 = X_2'X_1 = 0$$

$$\Rightarrow \quad \begin{pmatrix} X_1'X_1\hat{\beta}_1 \\ X_2'X_2\hat{\beta}_2 \end{pmatrix} = \begin{pmatrix} X_1'Y \\ X_2'Y \end{pmatrix}$$

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$$X_1'X_1\hat{\beta}_1 = X_1'Y$$

$$\Rightarrow \quad X_2'X_2\hat{\beta}_2 = X_2'Y$$

$$\hat{\beta}_1 = (X_1'X_1)^{-1}X_1'Y \qquad (5.37)$$

$$\Rightarrow \quad \hat{\beta}_2 = (X_2'X_2)^{-1}X_2'Y$$

provided $X_1'X_1$ and $X_2'X_2$ are non-singular

Thus $\hat{\beta}_1$ is the OLS estimate of β_1 irrespective of the fact whether $\hat{\beta}_2$ is so for β_2 or not. Now,

SS due to $\hat{\beta}_1 = \hat{\beta}_1'X_1'Y$

SS due to $\hat{\beta}_2 = \hat{\beta}_2'X_2'Y$

SS due to $(\hat{\beta}_1, \hat{\beta}_2) = \hat{\beta}'X'Y$

$$= (\hat{\beta}_1 \ \hat{\beta}_2)\begin{pmatrix} X_1'Y \\ X_2'Y \end{pmatrix}$$

$$= \hat{\beta}_1 X_1'Y + \hat{\beta}_2 X_2'Y$$

$$= SS \text{ due to } \beta_1 + SS \text{ due to } \beta_2 \qquad (5.38)$$

5.5.3 Partial *F*-test and Sequential *F*-tests

The method of Extra Sums of Squares (ESOS) provides a measure of extra sums of squares when one or more coefficients are introduced in the model, given the S.S due to other known coefficients. Suppose we introduce β_i in any sequence and obtain

$$SS(\hat{\beta}_i / \hat{\beta}_0, \hat{\beta}_1, ..., \hat{\beta}_{i-1}, \hat{\beta}_{i+1}, ..., \hat{\beta}_k); \ i = 1, 2, \ldots k$$

This sum of squares is distributed with one degree of freedom and the mean squares is same as sum of squares and can be tested by *F*-test against s^2 (the estimate of error variance σ^2). Now as a matter of fact, the additional sum of squares obtained by incorporating β_i depends on the sequence in which β_i has been introduced. If it is introduced first, it could have given most probably a higher sum of squares than if it is introduced

last, perhaps, the same can lead to much less sum of squares, because, the additional variables introduced (corresponding to the regression coefficient) might be correlated with the earlier variables and hence does not increase the sum of squares or increase the multiple correlation coefficient materially. We, while performing partial *F*-test, are interested to test the effect of additional sum of squares corresponding to the variables which is introduced last.

If the extra term under consideration is $\beta_i X_i$ we can say (of course approximately) in respect of a partial *F*-test on the variable X_i even though it is true that the partial *F*-test so performed is actually on the coefficient β_i.

Now the sequence in which a variable is introduced in a regression model is also an important item of consideration. For example, out of two variables X_1 and x_2 as explanatory variables for Y it may happen that x_1 may provide smaller predictive error than X_2, whereas X_2 may control Y more effectively than X_1. So the importance or the relative importance of the variables X_1 and X_2 while predicting Y must be judged from the purpose for which the explanatory variables are employed. When for a certain purpose variables are introduced one by one for performing *F*-test or Partial *F*-test, the test is known as ***Sequential F-test.***

5.5.4 Testing a General Linear Hypothesis in Regression Equation

Let us take the model

$$E(Y) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_k X_k$$

and test the following m-linear hypotheses, viz,

$$H_0 : C_{11}\beta_1 + C_{12}\beta_2 + \ldots + C_{1k}\beta_k = 0$$

$$C_{21}\beta_1 + C_{22}\beta_2 + \ldots + C_{2k}\beta_k = 0$$

$$\vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots$$

$$C_{m1}\beta_1 + C_{m2}\beta_2 + \ldots + C_{mk}\beta_k = 0 \qquad (5.39)$$

The above null hypothesis in matrix notation becomes

$$H_0 : C\beta = 0 \qquad (5.40)$$

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where $C = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1k} \\ C_{21} & C_{22} & \dots & C_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ C_{m1} & C_{m2} & \dots & C_{mk} \end{bmatrix}_{m \times k}$

and $\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix}_{k \times 1}$

Let $E(Y) = X\beta$

where $(Y)_{n\times 1} = (X)_{n\times k}(\beta)_{k\times 1} + (U)_{n\times 1}$ (5.41)

Then $\hat{\beta} = (X'X)^{-1}X'Y$ (5.42)

provided $(X'X)$ is non-singular. The sum of squares due to regression is given by

$Y'Y - \hat{\beta}X'Y$ with $(n-k)$ degrees of freedom (5.43)

Assuming that the null hypothesis in (5.40) represents a set of m-linear equations of which only q equations are independent and the rest of them *i.e.* $(k-q)$ variables are dependent on them. We have q independent equations to solve q of the k β's, viz., $\beta_1\ \beta_2 \dots \beta_q$ in terms of $\beta_{q+1}, \dots, \beta_k$. We get a reduced model of the form

$E(Y)_{n\times 1} = (Z)_{n\times k-q}(\alpha)_{k-q\times 1}$ (5.44)

We can now estimate the parameter vector

$\hat{\alpha} = (Z'Z)^{-1}Z'Y$ (5.45)

as usual provided $(Z'Z)$ is a non-singular matrix

S.S. due to regression $(SSR) = \hat{\alpha}'Z'Y$ (5.46)

Residual Sum of Squares $(SSW) = Y'Y - \hat{\alpha}'Z'Y$ (5.47)

with $n-(k-q) = n-k+q$ degrees of freedom

Since fewer degrees of freedom are involved in SSW than SSR,

$SSR = Y'Y - \hat{\beta}'X'Y$ (5.48)

It follows that SSW will be larger than SSR. Again SSW–SSR will represent the sum of squares due to hypothesis in (5.40) and the corresponding degrees of freedom will be

$n-k+q-(n-k) = n-k+q-n+k = q$ d.f.

Therefore, a test of hypothesis, viz, $H_0 : C\beta = 0$ is now made by considering the ratio

$$\frac{(SSW - SSR)/q}{SSR/n-k} \quad (5.49)$$

and the ratio is distributed as Snedecor's F with q and $(n-k)$ degrees of freedom respectively. This is an exact test under the assumption that the errors are independent and normally distributed.

Example 5.1 : Let $E(Y) = \beta_0 + \beta_1 X_1 + \beta_2 X_2$

Test $H_0 : \beta_1 = \beta_2 = 0$

Example 5.2 : $E(Y) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \dots + \beta_t X_t$

Test $H_0 : \beta_1 = \beta_2 = \dots = \beta_t = 0$,

t linear functions, all being independent.

Example 5.3 : $E(Y) = \beta_0 + \beta_1 X_1 + \dots + \beta_p X_p$

Test $H_0 : \beta_1 - \beta_2 = \beta_2 - \beta_3 = \dots = \beta_{p-1} - \beta_p = 0$

Hints : For examples 5.2 and 5.3 we can write the model as

$$E(Y) = X\beta_0 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{k\times 1} \beta_0$$

under H_0; $k = 2$ and t in examples 5.2 and 5;3 respectively.

Since $E(Y_i) = \beta_0;\ \forall\ i = 1, 2, \dots, k$ under H_0,

we have $\hat{\beta}_0 = \bar{y}$

$SSR = Y'Y - \hat{\beta}'X'Y$

with $n-k-1$ d.f.

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$$SSW = Y'Y - \bar{Y}(11\ldots1)\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$= Y'Y - n\bar{Y}^2 \text{ with } (n-1) \text{ d.f.}$$

$$SSW - SSR = Y'Y - n\,\bar{Y}^2 - Y'Y + \hat{\beta}'\,X'Y$$

$$= \hat{\beta}'\,X'Y - n\,\bar{Y}^2$$

with $(n-1)-(n-k-1) = k$ d.f.

Hence, the appropriate test is

$$F = \frac{\hat{\beta}'\,X'Y - n\,\bar{Y}^2/k}{Y'Y - \hat{\beta}'\,X'Y/(n-k-1)}$$

Example 5.4 : (An illustrative example)

Given $E(Y) = X\beta$ to be an appropriate model

test $H_0 : C\beta = 0$

where $Y' = (1, 5, 7, 9, 3, 7, 8);\ \beta' = (\beta_0, \beta_1, \beta_2, \beta_{11})$

$$X = \begin{bmatrix} 1 & X_1 & X_2 & X_1^2 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 93 \end{bmatrix}$$

$$(X'X)^{-1} = \begin{bmatrix} 7 & 0 & 3 & 4 \\ 0 & 4 & 0 & 0 \\ 3 & 0 & 9 & 0 \\ 4 & 0 & 0 & 4 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1/2 & 0 & -1/6 & -1/2 \\ 0 & 1/4 & 0 & 0 \\ -1/6 & 0 & 1/6 & 1/6 \\ -1/2 & 0 & 1/6 & 3/4 \end{bmatrix}$$

$$X'Y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \\ 7 \\ 9 \\ 3 \\ 7 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 40 \\ 6 \\ 33 \\ 22 \end{bmatrix}$$

$$\hat{\beta} = (X'X)^{-1}(X'Y)$$

$$= \begin{bmatrix} 1/2 & 0 & -1/6 & -1/2 \\ 0 & 1/4 & 0 & 0 \\ -1/6 & 0 & 1/6 & 1/6 \\ -1/2 & 0 & 1/6 & 3/4 \end{bmatrix}\begin{bmatrix} 40 \\ 6 \\ 33 \\ 22 \end{bmatrix}$$

$$= \begin{bmatrix} 7/2 \\ 3/2 \\ 5/2 \\ 2 \end{bmatrix}$$

$$\hat{\beta}'\,X'Y = (7/2\ \ 3/2\ \ 5/2\ \ 2)\begin{pmatrix} 40 \\ 6 \\ 33 \\ 22 \end{pmatrix} = 275.5$$

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$$Y'Y = (1\ 5\ 7\ 9\ 3\ 7\ 8)\begin{bmatrix}1\\5\\7\\9\\3\\7\\8\end{bmatrix}$$

$$= 278$$

$$SSR = Y'Y - \hat{\beta}'X'Y = 278 - 275.5 = 2.5$$

Therefore, the equation for the null hypothesis

$$H_0 : C\beta = 0$$

$$\beta_{11} = 0 \quad (i)$$

$$\beta_1 - \beta_2 = 0 \quad (ii)$$

$$\beta_1 - \beta_2 + \beta_{11} = 0 \quad (iii)$$

$$2\beta_1 - 2\beta_2 + 93\beta_{11} = 0 \quad (iv)$$

This hypothesis can be expressed in a much more simpler way as

$$H_0 : \beta_{11} = 0,\ \beta_1 = \beta_2 = \beta, \text{ say}$$

as the third and fourth equations are linear combinations of the first two equations. Now the linear model

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_{11} X_1^2$$

can be written as

$$Y = \beta_0 + \beta (X_1 + X_2)$$

$$= \alpha_0 + \alpha Z$$

where $\beta_0 = \alpha_0,\ \beta = \alpha$ and $Z = X_1 + X_2$

$$Z = \begin{bmatrix}1 & -2\\1 & 0\\1 & 0\\1 & 2\\1 & 0\\1 & 1\\1 & 2\end{bmatrix}$$

$$\Rightarrow \quad Z'Y = \begin{pmatrix}1&1&1&1&1&1&1\\-2&0&0&2&0&1&2\end{pmatrix}\begin{bmatrix}1\\5\\7\\9\\3\\7\\8\end{bmatrix}$$

$$= \begin{pmatrix}40\\39\end{pmatrix}$$

and $$(Z'Z)^{-1} = \begin{pmatrix}7&3\\3&13\end{pmatrix}^{-1}$$

$$= 1/82\begin{pmatrix}13&-3\\-3&7\end{pmatrix}$$

$$\hat{\alpha} = (Z'Z)^{-1}Z'Y$$

$$= 1/82\begin{pmatrix}13&-3\\-3&7\end{pmatrix}\begin{pmatrix}40\\39\end{pmatrix}$$

$$= 1/82\begin{pmatrix}403\\153\end{pmatrix}$$

$$\hat{\alpha}'Z'Y = 1/82\ (403\quad 153)\begin{pmatrix}40\\39\end{pmatrix}$$

$$= 269.3537$$

$$SSW = Y'Y - \hat{\alpha}'Z'Y = 8.6463$$

Now $p = 4,\ n = 7,\ q = 2,\ n - p = 3$

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$$SSW = 8.6463$$

$$SSR = 2.5$$

$$SSW - SSR = 5.8463 = \text{S.S due to hypothesis}$$

The appropriate test is now

$$F = \frac{(SSW - SSR)/q}{SSR/n - p}$$

$$= 3.5078$$

Since $F(2, 3, 0.95) = 9.55$ we do not reject H_0.

As the original model was

$$E(Y) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_{11} X_1^2$$

and the hypothesis is not rejected, it implies

$$\beta_1 = \beta_2 = \beta \text{ and } \beta_{11} = 0$$

Therefore, a more reasonable model would be

$$E(Y) = \beta_0 + \beta_1 (X_1 + X_2).$$

5.5.5 Weighted Least Squares

In 1.2 we have seen that under Gauss–Aitken set up, *i.e.*, under

$$Y = X\beta + U$$

with $D(Y) = \sigma^2 V$ instead of $D(Y) = \sigma^2 I$, the estimate

$$\tilde{\beta} = (X' V^{-1} X)^{-1} (X' V^{-1} Y)$$

and

$$D(\tilde{\beta}) = \sigma^2 (X' V^{-1} X)^{-1} \qquad (5.50)$$

The above results show that even if the assumptions of homosedasticity of the variance is violated, the simple least squares estimate remain unbiased for β. However, OLSE ceases to be the Minimum Variance Estimator as under the case of Gauss–Markov set up.

Example 5.5 : Let $Y = X\beta + U$

$$E(Y) = X\beta$$

and $D(U) = V\sigma^2$

$$= \begin{pmatrix} 1/\omega_1 & 0 & 0 & \dots & 0 \\ 0 & 1/\omega_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1/\omega_n \end{pmatrix} \sigma^2$$

$$= \text{diag}\,(1/\omega_1 \;\; 1/\omega_2 \dots \; 1/\omega_n)\,\sigma^2$$

$$\Rightarrow \quad V^{-1} = \text{diag}\,(\omega_1 \;\; \omega_2 \dots \; \omega_n)$$

$$\hat{\beta} = (X' V^{-1} X)^{-1} X' V^{-1} Y$$

$$= \left[(x_1 \; x_2 \dots \; x_n) \begin{pmatrix} \omega_1 & 0 & 0 & \dots & 0 \\ 0 & \omega_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \omega_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right]^{-1}$$

$$\times \left[(x_1 \; x_2 \dots \; x_n) \begin{pmatrix} \omega_1 & 0 & 0 & \dots & 0 \\ 0 & \omega_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \omega_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \right]$$

$$= \frac{\Sigma\, \omega_i x_i y_i}{\Sigma\, \omega_i x_i^2}, \quad i = 1, 2, \dots, n$$

Special Cases

Case I : Suppose $\text{Var}(y_i) = \lambda x_i$, *i.e.* $\text{Var}(y_i)$ is proportional to x_i; λ being the constant of proportionality.

$$\Rightarrow \quad w_i = \frac{\sigma^2}{\lambda x_i}$$

$$\hat{\beta} = \frac{\Sigma\, x_i y_i / \lambda x_i}{\Sigma\, x_i^2 / \lambda x_i} = \Sigma\, y_i / \Sigma\, x_i$$

$$= n\bar{y}/n\bar{x} = \bar{y}/\bar{x}$$

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i.e., the BLUE of β is $\bar{y}/\bar{x}$

Suppose Error of $\hat{\beta} = \sigma\sqrt{(X'V^{-1}X)^{-1}}$

$$= \sigma \sqrt{\left[(x_1\ x_2 \ldots x_n) \begin{pmatrix} \omega_1 & 0 & 0 & \ldots & 0 \\ 0 & \omega_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \omega_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right]^{-1}}$$

$$= \sqrt{\sigma^2 / \sum_{i=1}^{n} x_i^2 \omega_i} = \frac{\sigma}{\sqrt{\Sigma x_i^2 \frac{\sigma^2}{\lambda x_i}}} = \frac{\sqrt{\lambda}}{\sqrt{\Sigma x_i}}$$

$$\omega_i = \frac{\sigma^2}{\lambda x_i} \left(\text{as } \frac{\sigma^2}{\omega_i} = \lambda x_i \right)$$

Case II : Suppose Var $(y_i) = \lambda x_i^{\alpha} \Rightarrow \omega_i = \sigma^2 / \lambda x_i^{\alpha}$

BLUE of $$\beta = (X'V^{-1}X)^{-1}(XV^{-1}Y)$$

$$= \frac{\Sigma x_i y_i \omega_i}{\Sigma \omega_i x_i^2}$$

$$= \frac{\Sigma x_i y_i / \lambda x_i^{\alpha}}{\Sigma x_i^2 / \lambda x_i^{\alpha}}$$

$$= \Sigma y_i x_i^{1-\alpha} / \Sigma x_i^{2-\alpha}$$

and S.E. of $\hat{\beta} = \sqrt{\sigma^2 (X'V^{-1}X)^{-1}}$

$$= \sigma \sqrt{1/\Sigma\ \omega_i x_i^2}$$

$$= \sigma \sqrt{\frac{1}{\Sigma \sigma^2 x_i^2 / \lambda x_i^{\alpha}}}$$

$$= \sqrt{\lambda / \Sigma x_i^{2-\alpha}}$$

If $\alpha = 2$, BLUE of $\beta = \dfrac{\Sigma y_i / x_i}{n}$

and S.E. $(\hat{\beta}) = \lambda / n$.

5.5.6 Bias in the Regression Estimators on Account of Choice of Wrong Model

Suppose, we postulate a model

$$Y = X_1 \beta_1 + \epsilon_1$$

with $$E(\epsilon_1) = 0 \text{ and } D(\epsilon_1) = \sigma^2 I \qquad (5.51)$$

i.e. $$E(Y) = X_1 \beta_1$$

$$\hat{\beta}_1 = (X_1' X_1)^{-1} X_1' Y$$

If the postulated model is a correct one, then

$$E(\hat{\beta}_1) = (X_1' X_1)^{-1} X_1' E(Y)$$

$$= (X_1' X_1)^{-1} X_1' X_1 \beta_1 = \beta_1$$

$\Rightarrow \hat{\beta}_1$ is unbiased for β_1

Suppose the model in (5.51) is not the true relationship but

$$Y = X_1 \beta_1 + X_2 \beta_2 + \epsilon_2 \qquad (5.52)$$

with $$E(\epsilon_2) = 0 \text{ and } D(\epsilon_2) = \sigma^2 I$$

is the true model *i.e.* we did not incorporate the variable X_2 in the relationship. Then

$$E(\hat{\beta}_1) = (X_1' X_1)^{-1} X_1' E(Y)$$

$$= (X_1' X_1)^{-1} X_1' (X_1 \beta_1 + X_2 \beta_2)$$

$$= (X_1' X_1)^{-1} X_1' X_1 \beta_1 + (X_1' X_1)^{-1} X_1' X_2 \beta_2$$

$$= \beta_1 + (X_1' X_1)^{-1} X_1' X_2 \beta_2$$

This shows that $\hat{\beta}_1$ is biased for β_1, unless $X_1' X_2 = 0$. It may be noticed that the bias $(X_1' X_1)^{-1} X_1' X_2 \beta_2$ depend not only on the formulated and the true model but also on the design connected through the matrices X_1 and X_2. In other words, by a proper adjustment of the design it is possible to vary the level of bias even if the model chosen may wrong. This concept will be illustrated by the following examples:

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Example 5.6 : Suppose the formulated model is

$$E(Y) = \beta_0 + \beta_1 X$$

but the correct model is

$$E(Y) = \beta_0 + \beta_1 X + \beta_2 X^2$$

Let us consider the observations of Y corresponding to $X = -1, 0$ and 1. While estimating b_0 and b_1, we propose to examine the nature of bias in the estimates because of wrong formulation of the model.

We have

$$E(Y) = E\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\beta_2$$

$$= X_1 \beta_1 + X_2 \beta_2$$

$$(X_1' X_1)^{-1} = \left[\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}\right]^{-1}$$

$$= \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1/3 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$X_1' X_2 = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Thus we have

$$(X_1' X_1)^{-1} X_1' X_2 = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2/3 \\ 0 \end{pmatrix}$$

$$E\begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} + \begin{pmatrix} 2/3 \\ 0 \end{pmatrix}\beta_2$$

$$E(\hat{\beta}_0) = \beta_0 + 2\beta_2/3$$

$$E(\hat{\beta}_1) = \beta_1$$

Thus, $\hat{\beta}_0$ is biased although $\hat{\beta}_1$ is unbiased.

Example 5.7 : Suppose the formulated model is

$$E(Y) = \beta_0 + \beta_1 X$$

but the true model is given by

$$E(Y) = \beta_0 + \beta_1 X + \beta_2 X^2 + \beta_3 X^2$$

obtain the estimate of the bias in the parameter by taking observations

$$X = -3, -2, -1, 0, 1, 2, 3$$

$$X_1 = \begin{bmatrix} 1 & -3 \\ 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}; \quad X_2 = \begin{bmatrix} 9 & -27 \\ 4 & -8 \\ 1 & -1 \\ 0 & 0 \\ 1 & 1 \\ 4 & 8 \\ 9 & 27 \end{bmatrix}$$

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$$(X_1'X_1)^{-1} = \begin{pmatrix} 7 & 0 \\ 0 & 28 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1/7 & 0 \\ 0 & 1/28 \end{pmatrix}$$

$$X_1'X_2 = \begin{pmatrix} 28 & 0 \\ 0 & 196 \end{pmatrix}$$

$$(X_1'X_1)^{-1}X_1'X_2 = \begin{pmatrix} 4 & 0 \\ 0 & 7 \end{pmatrix}$$

$$\Rightarrow \quad E(\hat{\beta}_0) = \beta_0 + 4\beta_2$$

$$E(\hat{\beta}_1) = \beta_1 + 7\beta_3$$

The estimates are in general biased. The extent of the bias depends upon β_2 and β_3.

5.6 A Problem of Minimum Mean Square Prediction

Let $f(X) = E(Y|X)$ be the equation. Suppose Y is predicted by $f(\underset{\sim}{X})$. Mean Square Error M.S.E is given by $E[Y - f(\underset{\sim}{X})]^2$. The problem lies in minimising the mean square error w.r.t X. We can show that

$$f(\underset{\sim}{X}) = E(Y|\underset{\sim}{X}) = R(\underset{\sim}{X})$$

$$\text{M.S.E.} = E[Y - f(Y|\underset{\sim}{X}) + f(Y|\underset{\sim}{X}) - f(\underset{\sim}{X})]^2$$

$$= E[Y - f(Y|\underset{\sim}{X})^2] + E[f(Y|\underset{\sim}{X}) - f(\underset{\sim}{X})]^2$$

$$+ 2E[\{Y - f(Y|\underset{\sim}{X})\}\{f(Y|\underset{\sim}{X}) - f(\underset{\sim}{X})\}]$$

Also $E[\{Y - f(Y|\underset{\sim}{X})\}\{f(Y|\underset{\sim}{X}) - f(\underset{\sim}{X})\}]$

$$= E[E_x\{Y - f(Y|\underset{\sim}{X})\}\{f(Y|\underset{\sim}{X}) - f(\underset{\sim}{X})\}] = 0$$

as $E_x(Y - f(Y|\underset{\sim}{X})) = 0$

$$\Rightarrow \quad E[Y - f(\underset{\sim}{X})]^2 = E[Y - f(Y|\underset{\sim}{X})]^2 + E[f(Y|\underset{\sim}{X}) - f(\underset{\sim}{X})]^2$$

$\Rightarrow$ $E[Y - f(\underset{\sim}{X})]^2$ = M.S.E is minimum when

$$f(\underset{\sim}{X}) = f(Y|\underset{\sim}{X}) = R(\underset{\sim}{X})$$

5.6.1 Predictor Having Maximum Correlation Criterion

We have, $R(X) = f(Y|X)$, the regression of Y on X.

$0 \le \text{Corr}(\hat{Y}, Y = R(X)) \le 1$ is maximum

$$= \rho(Y, Y)$$

This is known as Multiple Correlation Coefficient

$$\rho_Y(X_1, X_2, \ldots, X_p)$$

Also $R(X) = f(Y|X_1, X_2, \ldots, X_p)$

We have

$$E[Y - E(Y)]^2 = E[Y - f(Y|X)]^2 + E[f(Y|X) - E(f(Y|X))]^2$$

$$\sigma_Y^2 = \sigma_{Y\underset{\sim}{X}}^2 + \sigma_{f(Y|\underset{\sim}{X})}^2$$

= variance due to departure from regression + variance due to regression

Correlation Ratio

$$= \eta_{Y\underset{\sim}{X}} = \frac{\sigma_{f(Y|\underset{\sim}{X})}}{\sigma_Y^2}$$

$$= \frac{\text{Variance due to regression}}{\text{Total variance}}$$

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Now η_{YX} may be employed in different choice of predictor variable or it provides a criterion for comparing different regression models for predicting Y.

5.6.2 Predictor of Production vis a vis Choice of Regression Model

Suppose we take the Cobb–Dauglas production function given by

$$P = cx_1^{\alpha} x_2^{\beta}$$

where P stands for production and x_1 and x_2, the volume of labour and capital inputs. α and β are the parameters which can be interpreted by elasticities of production with respect to labour and capital respectively (or shares of labour and capital respectively in the production process). Then

$$\log_e P = \log_e c + \alpha \log_e x_1 + \beta \log_e x_2$$

holds and one can regress $\log P$ on $\log x_1$ and $\log x_2$ simultaneously (although x_1 and x_2 seems to be correlated). The estimates of c, α and β can be obtained by solving the following normal equations formulated using the production, labour and capital investment data for a number of firms at a given time.

$$\left.\begin{aligned} \Sigma (\log P - \log c - \alpha \log x_1 - \beta \log x_2) &= 0 \\ \Sigma \log x_1 (\log P - \log c - \alpha \log x_1 - \beta \log x_2) &= 0 \\ \Sigma \log x_2 (\log P - \log c - \alpha \log x_1 - \beta \log x_2) &= 0 \end{aligned}\right\} \quad (5.53)$$

Now the Residual Variance is given by

$$\text{R.V} = [1/(n-2)] \Sigma (\log P - \log \hat{c} - \hat{\alpha} \log x_1 - \hat{\beta} \log x_2)^2 \quad (5.54)$$

and the total variance is

$$\text{T.V.} = (1/n) \Sigma (\log P - \log \bar{P})^2$$

where $\bar{P} = (\Sigma \log P)/n$

$1 - (\text{R.V}/\text{T.V})$ gives an estimate of $\eta_{P \mid x_1, x_2}$,

the correlation ratio of Production on Labour and Capital.

This can be compared with any other competing model predicting the production process with similar (or even dissimilar) inputs (explanatory variable) by estimating alternative $\eta_{p \mid \ldots}$. A comparison of such η's will give a clue for finding out the better of two regression models.

5.7 A Speciality for Linear Regression

The special case when the regression curve is linear has been studied extensively by Rao (1983). It follows that,

$\text{M.S.E} = E(Y - M)^2$ is minimum when $M = E(Y \mid X)$.

Let us consider an arbitrary linear function, say

$$\alpha + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_p X_p$$

and determine the coefficients $\alpha, \beta_1, \ldots, \beta_p$ by minimising the residual sum of squares viz.,

$$\begin{aligned} & E(Y - \alpha - \beta_1 X_1 - \beta_2 X_2 - \ldots - \beta_p X_p)^2 \\ &= E(Y^2 + \alpha^2 + \beta_1^2 X_1^2 + \beta_2^2 X_2^2 + \ldots + \beta_p^2 X_p^2) \\ &\quad - 2\alpha E(Y) - 2\beta_1 E(X_1 Y) - 2\beta_2 E(X_2 Y) - \ldots \\ &\quad - 2\beta_p E(X_p Y) + 2\alpha \beta_1 E(X_1) + 2\alpha \beta_2 E(X_2) + \ldots \\ &\quad + 2\alpha \beta_p E(X_p) + 2\beta_1 \beta_2 E(X_1 X_2) + \ldots + 2\beta_1 \beta_p E(X_1 X_p) \\ &\quad + \ldots + 2\beta_{p-1} \beta_p E(X_{p-1} X_p) \end{aligned} \quad (5.55)$$

Put $b = \alpha - E(Y) + \beta_1 E(X_1) + \beta_2 E(X_2) + \ldots + \beta_p E(X_p)$

Adding and subtracting b^2 to (5.55) we get

$$\begin{aligned} \text{R.S.S} = {} & \alpha^2 + [E(Y)]^2 + \beta_1^2 [E(X_1)]^2 + \beta_2^2 [E(X_2)]^2 + \ldots \\ & + \beta_p^2 E[(X_p)]^2 - 2\alpha E(Y) + 2\alpha \beta_1 E(X_1) + 2\alpha \beta_2 E(X_2) \\ & + \ldots + 2\alpha \beta_p E(X_p) - 2\beta_1 E(X_1) E(Y) - 2\beta_2 E(X_2) E(Y) \\ & - \ldots - 2\beta_p E(X_p) E(Y) + 2\beta_1 \beta_2 E(X_1) E(X_2) + \ldots \\ & + 2\beta_1 \beta_p E(X_1) E(X_p) + \ldots + 2\beta_{p-1} \beta_p E(X_{p-1}) E(X_p) \end{aligned}$$

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$$+\{E(Y^2)-[E(Y)]^2\}+\beta_1^2\{E(X_1^2)-[E(X_1)]^2\}$$
$$+\beta_2^2\{E(X_2^2)-[E(X_2)]^2\}+\ldots+\beta_p^2\{E(X_p^2)-E[(X_p)]^2\}$$
$$-2\beta_1\{E(X_1Y)-E(X_1)E(Y)\}-2\beta_p\{E(X_pY)-E(X_p)E(Y)\}$$
$$+\beta'C\beta-2\beta'\sigma_0+b^2;\ \beta=(\beta_1,\beta_2,\beta_p)'$$

where C is the dispersion matrix of $X_1, X_2, \ldots, X_p$ and σ_0 is the vector of covariance of Y with $X_1, X_2, \ldots, X_p$.

Thus we may write

$$E(Y-\alpha-\beta_1X_1-\beta_2X_2-\ldots-\beta_pX_p)^2$$
$$=b^2+\beta'C\beta-2\beta'\sigma_0+\text{Var}(Y) \qquad (5.56)$$

5.7.1 Optimum Choice for b and β given C, σ_0 and Var (Y)

Let b^* and β^* be the optimum choice of b and β respectively.

Put $\beta=\beta^*+\delta$, then we have

$$b^2+\beta'C\beta-2\beta'\sigma_0=b^2+(\beta^*+\delta)'C(\beta^*+\delta)-2(\beta^*+\delta)'\sigma_0$$
$$=b^2+\beta^{*\prime}C\beta^*+\delta'C\beta^*+\beta^{*\prime}C\delta$$
$$+\delta'C\delta-2\beta^{*\prime}\sigma_0-2\delta'\sigma_0$$

If we put $C\beta^*=\sigma_0$ and $b^*=0$ the R.H.S of above equation becomes

$$b^2+\beta^{*\prime}C\beta^*-2\beta^{*\prime}\sigma_0+\delta'C\delta\geq b^2+\beta^{*\prime}C\beta^*-2\beta^{*\prime}\sigma_0$$

(since $\delta'C\delta\geq 0$)

Also $\alpha^*=E(Y)-\beta^{*\prime}E(X)$, the optimum value of α

Therefore, the minimum value of

$$E(Y-\alpha-\beta_1X_1-\beta_2X_2-\ldots-\beta_pX_p)^2$$
$$=\text{Var}(Y)+\beta^{*\prime}C\beta^*-2\beta^{*\prime}\sigma_0$$

$$=\sigma_{00}-\sigma_0'C^{-1}\sigma_0$$

where $\sigma_{00}=\text{Var}(Y)$

Therefore, if the regression is linear, it can be completely specified by mean values, variances and covariances of the variables only and a knowledge of the exact distribution of variables is not necessary.

5.7.2 Test for the Equality of Regression Equations

We recall the problem of prediction from Cobb–Dauglas production function

$$p=cx_1^{\alpha}x_2^{\beta}$$
$$\log_e p=\log_e c+\alpha\log_e x_1+\beta\log_e x_2$$

Putting $\log_e p=Y$, $\log_e c=C$, $\log_e x_1=X_1$ and $\log_e x_2=X_2$

$$Y=C+aX_1+bX_2 \qquad (5.57)$$

is the linear relation between logarithms of production, labour and capital respectively. Suppose we fit the multiple regression model

$$Y=C^{(1)}+\alpha^{(1)}X_1+\beta^{(1)}X_2$$

on one series of data and with a different series of data

$$Y=C^{(2)}+\alpha^{(2)}X_1+\beta^{(2)}X_2$$

Under this circumstances we may opt to test two types of hypotheses, viz.,

(*i*) $C^{(1)}=C^{(2)}$, $\alpha^{(1)}=\alpha^{(2)}$, $\beta^{(1)}=\beta^{(2)}$

(*ii*) $\alpha^{(1)}=\alpha^{(2)}$, $\beta^{(1)}=\beta^{(2)}$;

irrespective of whether $C^{(1)}=C^{(2)}$ or not.

In case of (*i*) we say that the regression model is same for both the series while under (*ii*) we say that the two regression models are the same except for a change of origin. We proceed to develop a methodology fortesting both hypotheses. Let for the first and second series we denote the sample size and mean values as below :

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Sample size	Mean Values		
$n^{(1)}$	$\bar{y}_1^{(1)}$	$\bar{x}_1^{(2)}$	$\bar{x}_2^{(1)}$
$n^{(2)}$	$\bar{y}_1^{(2)}$	$\bar{x}_1^{(2)}$	$\bar{x}_2^{(2)}$

The defining the correct Sums of Squares and Sums of products as

$$S_{jk}^{(i)} = \sum_r (x_{jr}^{(i)} - \bar{x}_j^{(i)})(x_{rk}^{(i)} - \bar{x}_r^{(i)}),\ i = 1, 2$$

$$S_{0j}^{(i)} = \sum_r (x_{jr}^{(i)} - \bar{x}_j^{(i)})(y_r^{(i)} - \bar{y}^{(i)}) \text{ and}$$

$$S_{00}^{(i)} = \sum_r (y_r^{(i)} - \bar{y}^{(i)})^2$$

Now employing the above results, the regression equations

$$Y = C^{(i)} + \alpha^{(i)} X_1 + \beta^{(i)} X_2,\ i = 1, 2$$

are estimable.

The details of the testing procedure are given in pages 137-142 *(chapter 7)*.

6

APPLICATIONS OF LINEAR ESTIMATION TECHNIQUES

6.1 ANALYSIS OF VARIANCE (ANOVA)

Often a Researcher, while comparing whether or not there exists real difference in the measurements of a certain attribute in the two populations draws samples from each population. For example, he may wish to study a population of individuals with a certain disease who receive a standard treatment (or therapy) and another population of individuals with same disease, but receiving a new treatment. In this way he may plan to compare the effect of k treatments ($k > 2$). In such situations, the investigator, instead of using two samples at a time, may wish to make a single test to find out whether various treatments have the same effects. This kind of problems lead to application of F-test with single, double and multiple classifications; with one or more observations (equal or unequal) in a cell. This leads to the problem of Analysis of Variance (ANOVA) where we obtain estimates of variance relating to several components, viz., between patients, between treatments, between hospitals and also interaction (if any) between them. All ascribable factors having been identified and the sums of squares relating to all relavent factors having been worked out, the ratio of S.S due to treatments etc. are tested against

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S.S due to errors (i.e. residual S.S = Total S.S–S.S due to treatment) by F-test. The methodology is described in the next section.

6.1.1 Analysis of Variance with One Way Classification (Completely Randomised Design)

Let y_{ij} represent the value of the j^{th} unit with i^{th} treatment; $i = 1, 2, ..., k$; $j = 1, 2, ..., n_i$. Further let

$$E(y_{ij}) = \mu + \tau_i = \theta_i \tag{6.1}$$

where μ is the general mean and τ_i is the effect of the i^{th} treatment. Then

$$E(Y) = E(y_{11}, y_{12}, ..., y_{1n_1};\ y_{21}, ..., y_{2n_2};\ ...;\ y_{k1}, ..., y_{kn_k})'$$

$$= \alpha_1 \theta_1 + \alpha_2 \theta + ... + \alpha_\kappa \theta_\kappa \tag{6.2}$$

where $\alpha_i' = (0 \ldots 0 \quad 0 \ldots 0 \ldots 1 \ldots 1 \ldots 0 \ldots 0)$

$\qquad\qquad\qquad\quad n_1 \qquad n_2 \qquad n_i \qquad n_k$

We may write

$$E(Y) = (\alpha_1, \ \alpha_2 \ ..., \ \alpha_k) \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_k \end{bmatrix}$$

$$= A\,\theta \tag{6.3}$$

Obviously, rank of the estimation space $= \rho(V_a) = k$ (6.4)

or
$$\begin{bmatrix} \alpha_1' Y \\ \alpha_2' Y \\ \vdots \\ \alpha_k' Y \end{bmatrix} = \begin{bmatrix} \alpha_1'\alpha_1 & \alpha_1'\alpha_2 & \ldots & \alpha_1'\alpha_k \\ \alpha_2'\alpha_1 & \alpha_2'\alpha_2 & \ldots & \alpha_2'\alpha_k \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_k'\alpha_1 & \alpha_k'\alpha_2 & \ldots & \alpha_k'\alpha_k \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_k \end{bmatrix}$$

where $\alpha_i' \alpha_j = \delta_{ij} n_i$, $\forall\ i, j$ and δ_{ij} is Kronecker's delta

i.e. $\delta_{ij} = 1$ if $i = j$

$= 0$ if $i \neq j$

$$\begin{bmatrix} \alpha_1' Y \\ \alpha_2' Y \\ \vdots \\ \alpha_k' Y \end{bmatrix} = \begin{bmatrix} n_1 & 0 & \ldots & 0 \\ 0 & n_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & n_k \end{bmatrix} \begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \vdots \\ \hat{\theta}_k \end{bmatrix} \tag{6.5}$$

$$= \begin{bmatrix} n_1 \hat{\theta}_1 \\ n_2 \hat{\theta}_2 \\ \vdots \\ n_k \hat{\theta}_k \end{bmatrix}$$

Again $\alpha_1' Y = \sum_{j=1}^{n_1} y_{1j}$, $\alpha_2' Y = \sum_{j=1}^{n_2} y_{2j}$, ..., $\alpha_k' Y = \sum_{j=1}^{n_k} y_{kj}$

Thus normal equations are reduced to

$$\begin{bmatrix} \sum_{j=1}^{n_1} y_{ij} \\ \vdots \\ \sum_{j=1}^{n_k} y_{ij} \end{bmatrix} = \begin{bmatrix} n_1 \hat{\theta}_1 \\ n_2 \hat{\theta}_2 \\ \vdots \\ n_k \hat{\theta}_k \end{bmatrix}$$

$$\Rightarrow \quad \hat{\theta}_1 = \frac{1}{n_1} \sum_{j=1}^{n_1} y_{ij},\ \hat{\theta}_2 = \frac{1}{n_2} \sum_{j=1}^{n_2} y_{2j}, ..., \hat{\theta}_k = \frac{1}{n_k} \sum_{j=1}^{n_k} y_{kj} \tag{6.6}$$

Sum of Squares (S.S) due to estimates $= \hat{\Theta}' A' Y$

$$= \left[\frac{\sum_{j=1}^{n_1} y_{1j}}{n_1}, \frac{\sum_{j=1}^{n_2} y_{2j}}{n_2}, \ldots, \frac{\sum_{j=1}^{n_k} y_{kj}}{n_k} \right] \begin{bmatrix} \sum_{j=1}^{n_1} y_{ij} \\ \vdots \\ \sum_{j=1}^{n_k} y_{ij} \end{bmatrix}$$

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Denoting

$$\sum_{j=1}^{n_i} y_{ij} = Y_{i0} = T_i$$

$$\sum_{i=1}^{k} y_{ij} = Y_{0j} = T_j$$

and $$\sum_{ij} y_{ij} = Y_{00} = T_0 \qquad (6.7)$$

S.S due to estimates becomes

$$(Y_{10}/n_1 \quad Y_{20}/n_2 \quad \ldots \quad Y_{k0}/n_k) \begin{pmatrix} Y_{10} \\ \vdots \\ Y_{k0} \end{pmatrix}$$

$$= T_1^2/n_1 + T_2^2/n_2 + \ldots + T_k^2/n_k$$

$$= \sum_{i=1}^{k} T_i^2/n_i \qquad (6.8)$$

Testing of Hypothesis $H_0 : \theta_1 = \theta_2 = \ldots = \theta_k = \theta_0$, say

S.S due to estimates (under H_0) $= \Theta_0' A' Y$

Again from Normal equations (under H_0)

$$\begin{bmatrix} Y_{10} \\ Y_{20} \\ \vdots \\ Y_{k0} \end{bmatrix} = \begin{bmatrix} n_1 \hat{\theta}_0 \\ n_2 \hat{\theta}_0 \\ \vdots \\ n_k \hat{\theta}_0 \end{bmatrix} \qquad (6.9)$$

Denoting $Y_{00} = n \hat{\theta}_0$

or $\hat{\theta}_0 = Y_{00}/n$

$\hat{\Theta}_0 = (\hat{\theta}_0 \ \hat{\theta}_0 \ \ldots \ \hat{\theta}_0)'$ (6.10)

S.S due to estimates under H_0

$$= (Y_{00}/n \ \ldots \ Y_{00}/n)_{1 \times k} \begin{pmatrix} Y_{10} \\ \vdots \\ Y_{k0} \end{pmatrix}$$

$$= Y_{00}/n \ (Y_{10} + \ldots + Y_{k0})$$

$$= (Y_{00}/n) \ (Y_{00}) = (Y_{00})^2/n$$

$$= T^2/n$$

S.S due to deviation from hypothesis (6.11)

$$= \Theta' A' Y - \Theta_0' A' Y$$

$$= \sum_{i=1}^{k} T_i^2/n_i - T^2/n \qquad (6.12)$$

S.S due to errors (unrestricted) $= Y' Y - \Theta' A' Y$

$$= \sum_i \sum_j y_{ij}^2 - \sum_{i=1}^{k} T_i^2/n_i \qquad (6.13)$$

with $(n - k)$ degrees of freedom

S.S due to error under $H_0 = Y'Y - \Theta_0' A' Y$

$$= \sum \sum y_{ij}^2 - T^2/n \qquad (6.14)$$

with $(n - 1)$ d.f.

$\therefore$ S.S due to hypothesis

= S.S due to error under H_0 – S.S dueto error (unrestricted)

$$= \sum_{i=1}^{k} T_i^2/n_i - T^2/n \qquad (6.15)$$

with $(k - 1)$ d.f. which is same as (6.12). We can therefore construct ANOVA table for one way classification as follows:

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Source	d.f.	S.S	Mean Squares	F
Treatment	$k-1$	$\Sigma T_i^2/n_i - T^2/n = (i)$	$(i)/k-1 = \text{mst}$	$\frac{\text{mst}}{\text{mse}}$
Error	$n-k$	$\Sigma y_{ij}^2 - \Sigma T_i^2/n_i = (ii)$	$(ii)/n-k = \text{mse}$	
Total	$n-1$	$\Sigma y_{ij}^2 - T^2/n = (iii)$	$(iii)/n-1$	

Also $$\hat{\sigma}^2 = \frac{(Y - A\hat{\theta})'(Y - A\hat{\theta})}{n-k} \qquad (6.16)$$

6.1.2 ANOVA for Two Way Classification with One Observation Per Cell (Randomised Block Design)

Here we have the model

$$E(y_{ij}) = \mu + \alpha_i + \tau_j \qquad (6.17)$$

where μ = overall effect

α_i = effect of the i^{th} block

and τ_j = effect of the j^{th} treatment

in the context of a Randomised Block Design (RBD). In more detail we have,

$$E\begin{bmatrix} y_{11} \\ \vdots \\ y_{1v} \\ y_{21} \\ \vdots \\ y_{2v} \\ \vdots \\ y_{v1} \\ \vdots \\ y_{vv} \end{bmatrix} = m\mu + \sum_{i=1}^{b} b_i \alpha_i + \sum_{j=1}^{v} t_j \tau_j$$

where

$$m = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix}; \quad b_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \rightarrow 1^{st} \text{ block}; \quad \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \rightarrow i^{th} \text{ block}; \quad \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \rightarrow b^{th} \text{ block}$$

$$t_j = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} j^{th} \text{ treatment} \\ \text{in } 1^{st} \text{ block} \\ j^{th} \text{ treatment} \\ \text{in } i^{th} \text{ block} \\ j^{th} \text{ treatment} \\ \text{in } b^{th} \text{ block} \end{matrix}$$

$$\therefore \quad E(Y) = m\mu + \sum_{i=1}^{b} b_i \alpha_i + \sum_{j=1}^{v} t_j \tau_j$$

$$E(Y) = (m,\ b_1,\ b_2\ ...,\ b_b,\ t_1,\ t_2\ ...,\ t_v) \begin{bmatrix} \mu \\ \alpha_1 \\ \vdots \\ \alpha_b \\ \tau_1 \\ \vdots \\ \tau_v \end{bmatrix}$$

We can write $E(Y) = A\Theta$ (6.18)

where $A = (m, b_1\ b_2 \ldots b_b, t_1\ t_2 \ldots t_v)$

and $$\Theta = \begin{bmatrix} \mu \\ \alpha_1 \\ \vdots \\ \alpha_b \\ \tau_1 \\ \vdots \\ \tau_v \end{bmatrix}$$

Rank of the Estimation Space

$$\sum_{i=1}^{b} b_i = \sum_{j=1}^{v} t_j = m \qquad (6.19)$$

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and $$b_1 = \sum_{j=1}^{v} t_j - \sum_{i=2}^{b} b_i$$

$$\therefore \quad \rho(v_\alpha) \le b + v - 1 \tag{6.20}$$

If $b_2 \dots b_b$ and $t_1\, t_2 \dots t_v$ are linearly independent vectors, then

$$\sum_{i=2}^{b} c_i b_i + \sum_{j=1}^{v} d_j t_j = 0 \tag{6.21}$$

for constants $c_2 \dots c_b;\ d_1\, d_2 \dots d_v$

$$\Rightarrow \quad c_i = 0 \ (i = 2, 3, ..., b) \text{ and } d_j = 0 \ \forall\ j = 1, 2, ..., v$$

$$\text{Again (6.21)} \Rightarrow \begin{bmatrix} 0 \\ \vdots \\ 0 \\ c_2 \\ \vdots \\ c_2 \\ \vdots \\ c_b \\ \vdots \\ c_b \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_v \\ d_1 \\ \vdots \\ d_v \\ \vdots \\ d_1 \\ \vdots \\ d_v \end{bmatrix} = \begin{bmatrix} d_1 \\ \vdots \\ d_v \\ c_2 + d_1 \\ \vdots \\ c_2 + d_v \\ \vdots \\ c_b + d_1 \\ \vdots \\ c_b + d_v \end{bmatrix} = 0$$

This shows that

$$d_1 = d_2 = \dots = d_v = 0$$

which again gives

$$c_2 = c_3 = \dots = c_b = 0$$

Hence, from (6.21)

$$\rho(v_\alpha) = b + v - 1$$

Also if v_b = error space, then

$$\rho(v_b) = n - (b + v - 1)$$

$$= bv - b - v + 1$$

$$= (b-1)\ (v-1)$$

Also normal equations are given by (6.22)

$$A'Y = A'A\,\Theta$$

$$\Rightarrow \quad \begin{bmatrix} m' \\ b_1' \\ \vdots \\ b_b' \\ t_1' \\ \vdots \\ t_v' \end{bmatrix} Y$$

$$= \begin{bmatrix} m'm & m'b_1 & \dots & m'b_b & m't_1 & \dots & m't_v \\ b_1'm & b_1'b_1 & \dots & b_1'b_b & b_1't_1 & \dots & b_1't_v \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_b'm & b_b'b_1 & \dots & b_b'b_b & b_b't_1 & \dots & b_b't_v \\ b_b'm & b_b'b_1 & \dots & b_b'b_b & b_b't_1 & \dots & b_b't_v \\ t_1'm & t_1'b_1 & \dots & t_1'b_b & t_1't_1 & \dots & t_1't_v \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_v'm & t_v'b_1 & \dots & t_v'b_b & t_v't_1 & \dots & t_v't_v \end{bmatrix} \begin{bmatrix} \mu \\ \hat{\alpha}_1 \\ \vdots \\ \hat{\alpha}_b \\ \hat{\tau}_1 \\ \vdots \\ \hat{\tau}_v \end{bmatrix}$$

Also we can check from the following multiplication table

	m	b_i	t_j
m'	bv	v	b
b_i'	v	$\delta_{ii'}\, v$	1
t_j'	b	1	$\delta_{jj'} b$

(6.23)

where δ stands for Kronecker's delta. Also

$$m'Y = \sum_{ij} y_{ij} = Y_{00}$$

$$b_i'Y = \sum_{j} y_{ij} = Y_{i0}$$

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and $$t_j' Y = \sum_i y_{ij} = Y_{0j} \qquad (6.24)$$

Substituting the values in (6.23) and (6.24) in normal equations, we have

$$\begin{bmatrix} Y_{00} \\ Y_{10} \\ \vdots \\ Y_{b0} \\ Y_{01} \\ \vdots \\ Y_{0v} \end{bmatrix} = \begin{bmatrix} bv & (v\ v\ \dots\ v) & (b\ b\ \dots\ b) \\ \begin{pmatrix} v \\ \vdots \\ v \end{pmatrix} & \begin{pmatrix} v & 0 & \dots & 0 \\ 0 & v & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & v \end{pmatrix} & \begin{pmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \\ \begin{pmatrix} b \\ \vdots \\ b \end{pmatrix} & \begin{pmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} & \begin{pmatrix} b & 0 & \dots & 0 \\ 0 & b & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & b \end{pmatrix} \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{\alpha}_1 \\ \vdots \\ \hat{\alpha}_b \\ \hat{\tau}_1 \\ \vdots \\ \hat{\tau}_v \end{bmatrix}$$

$$\begin{bmatrix} Y_{00} \\ Y_{10} \\ \vdots \\ Y_{b0} \\ Y_{01} \\ \vdots \\ Y_{0v} \end{bmatrix} = \begin{bmatrix} nI & E_{1b}\ \text{diag}\ (v\ ..\ v) & E_{1v}\ \text{diag}\ (b\ ..\ b) \\ E_{1b}\ \text{diag}\ (v\ ..\ v) & \text{diag}\ (v\ ..\ v) & E_{bv} = N' \\ \text{diag}\ (b\ ..\ b)\ E_{v1} & E_{vb} = N & \text{diag}\ (b\ ..\ b) \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{\alpha}_1 \\ \vdots \\ \hat{\alpha}_b \\ \hat{\tau}_1 \\ \vdots \\ \hat{\tau}_v \end{bmatrix} \qquad (6.25)$$

Putting $$Y_{00} = G, \begin{bmatrix} Y_{10} \\ \vdots \\ Y_{b0} \end{bmatrix} = B, \begin{bmatrix} Y_{01} \\ \vdots \\ Y_{0v} \end{bmatrix} = T$$

$$\begin{bmatrix} \hat{\alpha}_1 \\ \vdots \\ \hat{\alpha}_b \end{bmatrix} = \hat{\alpha} \text{ and } \begin{bmatrix} \hat{\tau}_1 \\ \vdots \\ \hat{\tau}_v \end{bmatrix} = \hat{\tau}$$

We have the normal equations from (6.24) as

$$bv\hat{\mu} + v\sum_{i=1}^{b}\hat{\alpha}_i + b\sum_{j=1}^{v}\hat{\tau}_j = Y_{00} \qquad (6.26)$$

$$v\hat{\mu} + v\hat{\alpha}_i + \sum_{j=1}^{v}\hat{\tau}_j = Y_{i0} \qquad (6.27)$$

and $$b\hat{\mu} + \sum_{i=1}^{b}\hat{\alpha}_i + b\hat{\tau}_j = Y_{0j} \qquad (6.28)$$

In general, block and treatment contrasts are not estimable unless certain restrictions are imposed. If we apply the restrictions, say

$$\sum_i \hat{\alpha}_i = 0 \text{ and } \sum_j \hat{\tau}_j = 0 \qquad (6.29)$$

the normal equations are solvable. Such designs are known as ***connected designs.*** From (6.26) we have

$$\hat{\mu} = Y_{00}/bv = \bar{Y}_{00} \qquad (6.30)$$

From (6.27)

$$Y_{i0} = v\,(Y_{00}/bv) + v\,\hat{\alpha}_i$$

$\Rightarrow$ $$\hat{\alpha}_i = Y_{i0}/v - Y_{00}/bv,\ i = 1, 2, \dots, b$$

i.e. $$\hat{\alpha}_i = \bar{Y}_{i0} - \bar{Y}_{00} \qquad (6.31)$$

and from (6.28)

$$Y_{0j} = b\,(Y_{00}/bv) + b\,\hat{\tau}_j$$

$\Rightarrow$ $$\hat{\tau}_j = Y_{0j}/b - Y_{00}/bv$$

i.e. $$\hat{\tau}_j = \bar{Y}_{0j} - \bar{Y}_{00} \qquad (6.32)$$

Testing of Hypotheses

$$H_1 : \alpha_1 = \alpha_2 = \dots = \alpha_2$$

and $$H_2 : \tau_1 = \tau_2 = \dots = \tau_b$$

we proceed to obtain the S.S due to estimates

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$$\theta' A' Y = (\hat{\mu}\ .\ \hat{\alpha}_1 \ldots \hat{\alpha}_b, \hat{\tau}_1 \ldots \hat{\tau}_v) \begin{bmatrix} Y_{00} \\ Y_{10} \\ \vdots \\ Y_{b0} \\ Y_{01} \\ \vdots \\ Y_{0v} \end{bmatrix}$$

$$= \hat{\mu}\, Y_{00} + \sum_{i=1}^{b} \hat{\alpha}_i\, Y_{i0} + \sum_{j=1}^{v} \hat{\tau}_j\, Y_{0j}$$

$$= (Y_{00}/bv)\, Y_{00} + \sum_{i=1}^{b} (Y_{i0}/v - Y_{00}/bv)\, Y_{i0}$$

$$+ \sum_{j=1}^{v} (Y_{0j}/v - Y_{00}/bv)\, Y_{0j}$$

$$= Y_{00}^2/bv + \sum_{i=1}^{b} Y_{i0}^2/v - Y_{00}^2/bv$$

$$+ \sum_{j=1}^{v} Y_{0j}^2/b - Y_{00}^2/bv$$

$$= \sum_{i=1}^{b} Y_{i0}^2/v + \sum_{j=1}^{v} Y_{0j}^2/b - Y_{00}^2/bv \qquad (6.33)$$

with $(b+v-1)$ degrees of freedom, since rank of estimation space is $(b+v-1)$.

Again S.S due to estimates under H_1

$$= Y_{00}^2/bv + \sum_{j=1}^{v} Y_{0j}^2/b - Y_{00}^2/bv$$

$$= \sum_{j=1}^{v} Y_{0j}^2/b \qquad (6.34)$$

with v degrees of freedom.

S.S due to H_1 = S.S due to blocks

= S.S due to estimates (unrestricted)

− S.S due to estimates under H_1

= (6.33) − (6.34)

$$= \sum_{i=1}^{b} Y_{i0}2/v - Y_{00}^2/bv \qquad (6.35)$$

with $(b+v-1)-v = b-1$ degrees of freedom.

S.S due to estimates under $H_2 = Y_{00}^2/bv + \sum_{i=1}^{b} Y_{i0}^2/v - Y_{00}^2/bv$

$$= \sum_{i=1}^{b} Y_{i0}^2/v \text{ with } b \text{ degrees of freedom} \qquad (6.36)$$

S.S due to H_2 = S.S due to treatments

= S.S due to estimates (unrestricted)

− S.S due to estimates under H_2

= (6.33) − (6.36)

$$= \sum_{j=1}^{v} Y_{0j}^2/b - Y_{00}^2/bv \qquad (6.37)$$

with $(b+v-1)-b = v-1$ degrees of freedom

Finally, S.S due to error (unrestricted) = $Y'Y - \hat{\Theta}' A' Y$

where $\hat{\Theta}' A' Y$ is the S.S due to estimates (unrestricted) and

$$Y'Y = \sum_{i,j} Y_{ij}^2 - \sum_{i} \frac{Y_{i0}^2}{v} - \sum_{j} \frac{Y_{0j}^2}{b} + Y_{00}^2/bv \qquad (6.38)$$

is the total sum of squares with $bv - b - v + 1 = (b-1)(v-1)$ d.f.

Hence, by adding up, total sum of squares (T.S.S)

$$= \sum_{i,j} Y_{ij}^2 - Y_{00}^2/bv \qquad (6.39)$$

with $(bv-1)$ degrees of freedom.

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ANOVA TABLE

Source	d.f.	Sum of Squares	Mean Square	F
Block	$(b-1)$	$\frac{\Sigma Y_{i0}^2}{v} - \frac{Y_{(00)}^2}{bv} = \text{SSB}$	$\frac{\text{SSB}}{(b-1)} = \text{MSB}$	$\frac{\text{MSB}}{\text{MSE}}$
Treatment	$(v-1)$	$\frac{\Sigma Y_{0j}^2}{b} - \frac{Y_{(00)}^2}{bv} = \text{SST}$	$\frac{\text{SST}}{(v-1)} = \text{MST}$	$\frac{\text{MST}}{\text{MSE}}$
Error	$(b-1)(v-1)$	$\Sigma Y_{ij}^2 - \frac{\Sigma Y_{i0}^2}{v} - \frac{\Sigma Y_{0j}^2}{b} + \frac{Y_{(00)}^2}{bv} = \text{SSE}$	$\frac{\text{SSE}}{(b-1)(v-1)}$	
Total	$(bv-1)$	$\Sigma Y_{ij}^2 - \frac{Y_{(00)}^2}{bv} = \text{TSS}$		

6.1.3 Two Way Classification with Equal Number of Observations (say, r) per Cell (Replicated Randomised Block Design)

We consider

$$E(y_{ijl}) = \mu + \alpha_i + \tau_j + \gamma_{ij}$$

where

$\mu \equiv$ overall effect

$\alpha_i \equiv$ effect of the i^{th} block; $i = 1, 2, \ldots, b$

$\tau_j =$ effect of the j^{th} treatment; $j = 1, 2, \ldots, v$

As usual we can write the model as

$$E(Y) = m\mu + \Sigma\, b_i \alpha_i + \Sigma\, b_j \tau_j + \Sigma\, c_{ij} \gamma_{ij} \qquad (6.40)$$

where

$$Y = \begin{bmatrix} y_{111} \\ \vdots \\ y_{11r} \\ y_{211} \\ \vdots \\ y_{22r} \\ \vdots \\ y_{bv1} \\ \vdots \\ y_{bvr} \end{bmatrix}_{bvr \times 1} ; \; m = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}_{bvr \times 1} ; \; b_i = \begin{bmatrix} 0 \\ \vdots \\ 1\,(i,1) \\ \vdots \\ 1\,(i,i) \\ \vdots \\ 1 \\ 1\,(i,v) \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$$t_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix}(i,j) \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix}(2,j) \\ \vdots \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix}(b,j) \rightarrow \\ 0 \\ \vdots \\ 0 \end{bmatrix} ; \qquad c_{ij} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \\ 1 \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{matrix} \left.\begin{matrix} \\ \\ \\ \\ \\ : (i,j) \\ \\ \\ \\ \\ \end{matrix}\right\} \rightarrow \begin{matrix} r \text{ units of} \\ i^{th} \text{ block} \\ \text{and } j^{th} \\ \text{treatment} \end{matrix}$$

br units corresponding to j^{th} treatment

Thus we have $E(Y) = A\,\Theta$

where $A = (m, b_1, \ldots, b_b, t_1, \ldots, t_v, c_{11}, \ldots c_{bv})$ and

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$$\Theta = \begin{bmatrix} \mu \\ \alpha_1 \\ \vdots \\ \alpha_b \\ \tau_1 \\ \vdots \\ \tau_v \\ \gamma_{11} \\ \vdots \\ \gamma_{bv} \end{bmatrix}_{(bv+b+v+1)\times 1}$$

Rank of the Estimation Space

We note

$$\sum_{j=1}^{v} c_{ij} = b, \quad \sum_{i=1}^{b} c_{ij} = v \text{ and } \sum_{i,j} c_{ij} = bvr \tag{6.41}$$

It follows that first $(b+v+1)$ vectors in A given above can be expressed as a linear combination of the last bv vectors

$$\therefore \quad \rho(v_\alpha) \le bv \tag{6.42}$$

Now among bv vectors c_{ij} are mutually orthogonal to each other

Also $\quad c_{i'j'} c_{ij} = \delta_{ii'} \delta_{jj'} r$

$$\therefore \quad \rho(v_\alpha) = bv \tag{6.43}$$

We have the following scheme

	m	b_i	t_j	c_{ij}
m'	bvr	vr	br	r
b_i'	vr	$\delta ii' vr$	r	$\delta_{ii'} r$
t_j'	br	r	$\delta_{jj'} br$	$\delta_{jj'} r$
$c_{i'j}$	r	$\delta ii' r$	$\delta_{jj'} r$	$\delta_{ii'} \delta_{jj'} r$

The normal equations are given by

$$A'Y = A'A\,\Theta$$

$$\begin{bmatrix} Y_{000} \\ Y_{100} \\ \vdots \\ Y_{b00} \\ Y_{010} \\ \vdots \\ Y_{0v0} \\ Y_{110} \\ \vdots \\ Y_{bv0} \end{bmatrix} = \begin{bmatrix} bvr & (vr \dots vr) & (br \dots br) & (r\ r \dots r\ r) \\ vr \\ \vdots \\ vr \\ br \\ \vdots \\ br \\ r \\ \vdots \\ r \end{bmatrix}$$

$$\times (\hat{\mu}\ \hat{\alpha}_1 \dots \hat{\alpha}_b\ \hat{\tau}_1 \dots \hat{\tau}_v\ \hat{\gamma}_{11} \dots \hat{\gamma}_{bv})'$$

Hence, the normal equations can be written as

$$Y_{000} = bvr\hat{\mu} + vr \sum_{i=1}^{b} \hat{\alpha}_i + br \sum_{j=1}^{v} \hat{\tau}_j + r \sum_{i,j} \hat{\gamma}_{ij} \tag{6.44}$$

$$Y_{i00} = vr\hat{\mu} + vr\hat{\alpha}_i + r \sum_{j=1}^{v} \hat{\tau}_j + r \sum_{j=1}^{v} \hat{\gamma}_{ij} \tag{6.45}$$

$$Y_{0j0} = br\hat{\mu} + r \sum_{i=1}^{b} \hat{\alpha}_i + br\hat{\tau}_j + r \sum_{i=1}^{b} \hat{\gamma}_{ij} \tag{6.46}$$

We have, therefore,

$$Y_{ij0} = r\hat{\mu} + r\hat{\alpha}_i + r\hat{\tau}_j + r\hat{\gamma}_{ij} \tag{6.47}$$

to test the hypotheses

$$\left.\begin{aligned} H_1 &: \alpha_1 = \alpha_2 = \dots = \alpha_b \\ H_2 &: \tau_1 = \tau_2 = \dots = \tau_v \\ H_3 &: \gamma_{11} = \gamma_{12} = \dots = \gamma_{bv} \end{aligned}\right\} \tag{6.48}$$

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In general, block, treatment and interaction contrasts are not estimable. But if we impose the restrictions, viz., $\sum_j \gamma_{ij} =$ constant $\forall\ i$ and $\sum_i \gamma_{ij} =$ constant $\forall\ j$, then we show that all contrasts are estimable.

Block Contrasts

Let us first try to estimate $(\alpha_i - \alpha_{i'})$ for $i \neq i'$. From (6.44), we have,

$$Y_{i00} = vr\hat{\mu} + vr\hat{\alpha}_i + r\sum_{j=1}^{v}\hat{\tau}_j + r\sum_{i,j}\hat{\gamma}_{ij}$$

$$Y_{i00} - Y_{i'00} = vr(\hat{\alpha}_i - \hat{\alpha}_{i'}) + r\left(\sum_{i,j}^{v}\hat{\gamma}_{ij} - \sum_{i,j}^{v}\hat{\gamma}_{i,j}\right)$$

But given $\sum \gamma_{ij} =$ constant

$$\Rightarrow \quad \left(\sum_{i,j}\hat{\gamma}_{ij} - \sum_{i,j}\hat{\gamma}_{ij}\right) = 0$$

$$\therefore \quad (Y_{i00} - Y_{i'00})/vr = (\hat{\alpha}_i - \hat{\alpha}_{i'})$$

$$\Rightarrow \quad \bar{Y}_{i00} - \bar{Y}_{i'00}) = (\hat{\alpha}_i - \hat{\alpha}_{i'}) \tag{6.49}$$

Since any other block contrasts are capable of being expressed as a linear combination of $\alpha_i - \alpha_{i'}$ for different i and i', every block contrasts is estimable under the given restrictions.

Treatment Contrasts

We have from (6.46)

$$Y_{0j0} = br\hat{\mu} + r\sum_{i=1}^{b}\hat{\alpha}_i + br\hat{\tau}_j + r\sum_{i=1}^{b}\hat{\gamma}_{ij}$$

$$Y_{0j'0} = br\hat{\mu} + r\sum_{i=1}^{b}\hat{\alpha}_i + br\hat{\tau}_{j'} + r\sum_{i=1}^{b}\hat{\gamma}_{ij'}$$

$$\therefore \quad Y_{0j0} - Y_{0j'0} = br(\hat{\tau}_j - \hat{\tau}_{j'}) + r\left(\sum_{i=1}^{b}\hat{\gamma}_{ij} - \sum_{i=1}^{b}\hat{\gamma}_{ij'}\right)$$

But $\sum_i \gamma_{ij} =$ constant

$$\Rightarrow \quad \left(\sum_{i=1}^{v}\hat{\gamma}_{ij} - \sum_{i=1}^{v}\hat{\gamma}_{ij'}\right) = 0$$

$$\therefore \quad (Y_{0j0} - Y_{0j'0})/br = (\hat{\tau}_j - \hat{\tau}_{j'})$$

$$\Rightarrow \quad \bar{Y}_{0j0} - \bar{Y}_{0j'0}) = (\hat{\tau}_j - \hat{\tau}_{j'}) \tag{6.50}$$

In other words $\tau_j - \tau_{j'}$ is estimable by $\bar{Y}_{0j0} - \bar{Y}_{0j'0}$. But the preceding argument, every treatment contrast can be estimated under given restrictions.

Interaction Contrasts

From (6.47) we have

$$Y_{ij0} = r\hat{\mu} + r\hat{\alpha}_i + r\hat{\tau}_j + r\hat{\gamma}_{ij}$$

$$Y_{i'j'0} = r\hat{\mu} + r\hat{\alpha}_{i'} + r\hat{\tau}_{j'} + r\hat{\gamma}_{i'j'}$$

$$Y_{ij0} - Y_{i'j'0} = r(\hat{\alpha}_i - \hat{\alpha}_{i'}) + r(\hat{\tau}_j - \hat{\tau}_{j'}) + r(\hat{\gamma}_{ij} - \hat{\gamma}_{i'j'})$$

It is obviously seen that the interaction contrast is not estimable by $\bar{Y}_{ij0} - \bar{Y}_{i'j'0}$. Also

$$Y_{ij0} - Y_{i'j'0} = r(\bar{Y}_{i00} - \bar{Y}_{i'00}) + r(\bar{Y}_{0j0} - \bar{Y}_{0j'0}) + r(\hat{\gamma}_{ij} - \hat{\gamma}_{i'j'})$$

$$\Rightarrow \quad \bar{Y}_{ij0} - \bar{Y}_{i'j'0} = (\bar{Y}_{i00} - \bar{Y}_{i'00}) + (\bar{Y}_{0j0} - \bar{Y}_{0j'0}) + (\hat{\gamma}_{ij} - \hat{\gamma}_{i'j'})$$

or

$$(\hat{\gamma}_{ij} - \hat{\gamma}_{i'j'}) = (\bar{Y}_{ij0} - \bar{Y}_{i'j'0}) - (\bar{Y}_{i00} - \bar{Y}_{i'00}) - (\bar{Y}_{0j0} - \bar{Y}_{0j'0}) \tag{6.51}$$

By a similar argument, it follows that interaction contrast is estimable under above restriction.

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Now S.S due to estimates (unrestricted) $= \hat{\Theta}' A' Y$

$$= \begin{bmatrix} \hat{\mu} \\ \hat{\alpha}_1 \\ \vdots \\ \hat{\alpha}_b \\ \hat{\tau}_1 \\ \vdots \\ \hat{\tau}_v \\ \hat{\gamma}_{11} \\ \vdots \\ \hat{\gamma}_{bv} \end{bmatrix} (Y_{000}\ Y_{100} \ \dots\ Y_{b00}\ Y_{010} \ \dots\ Y_{0v0}\ Y_{110} \ \dots\ Y_{bv0})$$

$$= \hat{\mu}\, Y_{000} + \sum_i \hat{\alpha}_i\, Y_{i00} + \sum_j \hat{\tau}_j\, Y_{0j0} + \sum_{i,j} \hat{\gamma}_{ij}\, Y_{ij0}$$

$$= \hat{\mu} \sum_{i,j} Y_{ij0} + \sum_{i,j} \hat{\alpha}_i\, Y_{ij0} + \sum_{i,j} \hat{\tau}_j\, Y_{ij0} + \sum_{i,j} \hat{\gamma}_{ij}\, Y_{ij0}$$

$$= \sum_{i,j} (\hat{\mu} + \hat{\alpha}_i + \hat{\tau}_j + \hat{\gamma}_{ij})\, Y_{ij0}$$

$$= \sum_{i,j} (Y_{ij0}/r)\, Y_{ij0}$$

$\therefore$ S.S due to estimates $= \sum_{i,j} (Y_{ij0}^2/r)$ (6.52)

with bv degrees of freedom, because $\rho(A) = bv$.

S.S due to error (unrestricted)

$= Y'Y - \hat{\Theta}' A' Y$ with $(bvr - bv) = bv(r-1)$ d.f.

$$= \sum_{i,j,l} y_{ijl}^2 - \sum_{i,j} Y_{ij0}^2/r \text{ with } bv(r-1) \text{ d.f.} \quad (6.53)$$

To test interaction, $H_3 : (\gamma_{11} = \gamma_{12} = \dots = \gamma_{bv} = c$, say) we have,

$$E(Y) = m\mu + \sum_{i=1}^{b} b_i \alpha_i + \sum_{j=1}^{v} t_j \tau_j + \sum_{i,j} c_{ij} \gamma_{ij}$$

under H_3,

$$E(Y) = m\mu + \sum_{i=1}^{b} b_i \alpha_i + \sum_{j=1}^{v} t_j \tau_j + \text{cm} \quad \left(\text{as } \sum_{i,j} c_{ij} = c\right)$$

$$= m(c + \mu) + \sum_{i=1}^{b} b_i \alpha_i + \sum_{j=1}^{v} t_j \tau_j$$

Note that this model becomes the same as that of Randomised Block Design (RBD) with one observation per cell.

S.S due to estimates under H_3 = S.S due to estimates (unrestricted) under R.B.D with r observations per cell

$$= \sum_j Y_{i00}^2/vr + \sum_j Y_{0j0}^2/br - Y_{000}^2/bvr \quad (6.54)$$

S.S due to interaction = S.S due to H_3

$$= \sum_{i,j} Y_{ij0}^2/r - \sum_i Y_{i00}^2/vr - \sum_j Y_{0j0}^2/br + Y_{000}^2/bvr \quad (6.55)$$

with $(bv - b - v + 1) = (b-1)(v-1)$ d.f. [as S.S due to hypothesis = S.S due to estimates (unrestricted) – S.S due to estimates under the hypothesis.]

S.S due to estimates – S.S due to estimates under H_1

= S.S due to H_1

$\therefore$ S.S due to estimates (under H_3) – S.S due to estimates under H_1 (given H_3 to be true)

= S.S due to H_1 (given H_3 to be true)

$$= \sum_i \frac{Y_{i00}^2}{vr} + \sum_j \frac{Y_{ij0}^2}{br} - \sum_j \frac{Y_{0j0}^2}{br} - \frac{Y_{000}^2}{bvr}$$

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$$= \sum_i \frac{Y_{i00}^2}{vr} - \frac{Y_{000}^2}{bvr}, \text{ with } (b+v-1)-v=(b-1) \text{ d.f.}$$

$$= S.S \text{ due to blocks under } H_3 \tag{6.56}$$

Precisely, by the same way;

S.S due to estimates under H_3 – S.S due to estimates under H_2 (given H_3 to be true)

$$= S.S \text{ due to } H_2 \text{ when } H_3 \text{ is true}$$

$$= \sum_i \frac{Y_{i00}^2}{vr} + \sum_j \frac{Y_{0j0}^2}{br} - \sum_i \frac{Y_{i00}^2}{br} - \frac{Y_{000}^2}{bvr}$$

$$= \sum_i \frac{Y_{0j0}^2}{vr} - \frac{Y_{000}^2}{bvr}, \text{ with } (b+v-1)-b=(v-1) \text{ d.f.}$$

$$= S.S \text{ due to treatments under } H_3 \tag{6.57}$$

Now under H_3 the usual additivity of the S.S holds

$\therefore$ from (6.53), (6.55), (6.56) and (6.57)

$$\text{Total} \qquad S.S = \sum_{i,j,l} Y_{ijl}^2 - \frac{Y_{000}^2}{bvr} = T \tag{6.58}$$

with $(bvr-1)=(n-1)$ d.f.

6.1.4 Some Critical Questions on the Appropriate Tests of ANOVA

(*i*) Is the interaction significant ?

The appropriate test will be $F = V_{AB}/V_E$ with $(b-1)$ $(v-1)$ and $bv(r-1)$ d.f. respectively.

(*ii*) Suppose the interaction component is found significant. Is there difference between blocks averaged over all given treatments even when there may be difference between blocks for a particular treatment ?

Here we require to test blocks with fixed treatment;

e.g. $\alpha_{1i} = \alpha_{2i} = \ldots = \alpha_{vi}$

The appropriate test is V_A/V_E.

(*iii*) Is there difference between treatments averaged over all given blocks even when there is a difference between treatments for a particular block ?

The appropriate test is V_B/V_E.

(*iv*) Is there difference between blocks, averaged over all treatments in a population of which the given treatments in the experiment constitute a random sample only ?

The appropriate test is V_A/V_{AB}.

(*v*) Is there difference between treatments arranged over all blocks in the population of which the blocks included in the experiment constitute a random sample only ?

The appropriate test is V_B/V_{AB}.

Note : The difference between [(*ii*) and (*iv*)] or [(*iii*) and (*v*)] is important to be noted. In case there is no interaction, note that [(*ii*) and (*iv*)] or [(*iii*) and (*v*)] are the same.

6.2 Analysis of Covariance (ANCOVA)

The technique of ANOVA enables us to consider the factors qualitatively while Regression analysis technique is restricted to treat the factors quantitatively. Analysis of Covariance (ANCOVA) is considered to be an extension of ANOVA technique in the sense that here we consider some of the factors quantitatively while others are dealt qualitatively. The variables which are treated quantitatively are called ***Concomittant*** variables.

Sometimes it happens that even after taking into account of all the plausible qualitative factors in a design, the residual variance still remains significantly higher. For example, the variance of the treatment contrast, say $\text{Var}(\tau_i - \tau_{i'})$ which otherwise could have been highly significant shows insignificance when tested against the error variance which is often very large. This makes the design inefficient.

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Hence, we require to introduce certain other factors which could be dealt quantitatively to bring the residual variation. The technique by which the analysis is performed is known as **Analysis of Covariance (ANCOVA).**

6.2.1 The Problem of Choosing Concomittant Variables

The choice of Concomittant variables should be governed by the following considerations.

(i) The Concomittant variables $x_1, x_2, ..., x_p$ should have a direct cause and effect relationship (not merely the correlation) with Y, the dependent variable. The effect of association (linear) may be measured by Multiple Correlation coefficient of Y on $x_1, x_2, ..., x_p$, i.e. $R_{Y(12\ldots p)}$. However, a high correlation does not necessarily mean that the variables $x_1, x_2, ..., x_p$ should be considered Concomittant.

(ii) The Concomittant variables (each of $x_1, x_2, ..., x_p$) should be independent of those factors which are treated qualitatively.

(iii) $x_1, x_2, ..., x_p$ should be independent among themselves and they should be uncorrelated with the error component, say $\in_{ijk} \ldots$ in a linear model.

(iv) It is advantageous to assume $D(Y) = \sigma^2 I$ or y_i's (the observations on the r.v., Y) are mutually independent and have constant variance σ^2.

(v) It is also advantageous and which is indeed a practice to take x_i's to be nonstochastic.

6.2.2 ANCOVA for RBD with *p* Concomittant Variables

Consider the model

$$E(y_{ij}) = \mu + \alpha_i + \tau_j + \xi_1 (x_{1(ij)} - \bar{x}_{1(00)}) + \xi_2 (\bar{x}_{2(ij)} - x_{2(00)}) + \ldots + \xi_p (\bar{x}_{p(ij)} - x_{p(00)}) + \in_{ij}$$

$$\Rightarrow \quad E(Y) = m\mu + \sum_{i=1}^{b} b_i \alpha_i + \sum_{j=1}^{v} t_j \tau_j + \sum_{r=1}^{p} \xi_r C_r$$

where

$$m = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{bv \times 1} \qquad b_i = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow i^{th} \text{ unit corresponding to } v \text{ unities}$$

$$t_j = \begin{bmatrix} 0 \\ 1 \rightarrow \\ 1 \\ 0 \\ \vdots \\ 0 \\ 1 \rightarrow \\ 0 \end{bmatrix} \begin{matrix} \\ j^{th} \text{ treatment corresponding to unities} \\ \\ \\ \\ \\ j^{th} \text{ treatment corresponding to unities} \\ \end{matrix} \qquad C_r = \begin{bmatrix} x_{r(11)} - \bar{x}_{r(00)} \\ x_{r(12)} - \bar{x}_{r(00)} \\ \ldots \ \ldots \ \ldots \ \ldots \\ x_{r(bv)} - \bar{x}_{r(00)} \end{bmatrix}$$

where $x_{r(00)} = \frac{1}{bv} \sum_{i,j} x_{r(ij)}$

Hence, we write the model as $E(Y) = A\Theta$

where $A = [m, b_1, b_2, \ldots, b_b, t_1, t_2 \ldots t_v, c_1, c_2, ..., c_p]$

$$\Theta = \begin{bmatrix} \mu \\ \alpha_1 \\ \vdots \\ \alpha_p \\ \tau_1 \\ \vdots \\ \tau_v \\ \xi_1 \\ \vdots \\ \xi_p \end{bmatrix}$$

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where ξ_r, $r = 1, 2, ..., b$ are partial regression coefficients. Symbols are denoted as below:

Source	S.S (Y)	S.P. Matrix of $(x_1, x_2, ..., x_p)$ and y	S.P. Matrix of $x_1, x_2, ..., x_p$
Block	B_{yy}	$B_{1y}\ B_{2y}\ \dots\ B_{py}$	$(B_{rs})_{p\times p}$
Treatment	T_{yy}	$T_{1y}\ T_{2y}\ \dots\ T_{py}$	$(T_{rs})_{p\times p}$
Error	E_{yy}	$E_{1y}\ E_{2y}\ \dots\ E_{py}$	$(E_{rs})_{p\times p}$
Total	G_{yy}	$G_{1y}\ G_{2y}\ \dots\ G_{py}$	$(G_{rs})_{p\times p}$

Here, $B_{ry} = v \sum_{i=1}^{b} (\bar{x}_{(r)\,i0} - \bar{x}_{r\,(00)})\,(\bar{y}_{i0} - \bar{y}_{00})$

$$T_{sy} = v \sum_{j=1}^{v} (x_{(s)\,ij} - \bar{x}_{(s)\,00})\,(\bar{y}_{0j} - \bar{y}_{00});\ s = 1, 2, ..., p$$

$$G_{ry} = \sum_{i} \sum_{j} (x_{(r)\,ij} - \bar{x}_{(r)\,00})\,(y_{ij} - \bar{y}_{00})$$

$$G_{sy} - B_{sy} - T_{sy} = E_{sy};\ s = 1, 2, ..., p$$

$(B_{rs})_{p\times p}$ is defined by

$$B_{rs} = v \sum_{i=1}^{b} (\bar{x}_{(r)\,i0} - \bar{x}_{r\,(00)})\,(\bar{x}_{(s)\,i0} - \bar{x}_{(s)\,00})$$

Similarly, $(T_{rs})_{p\times p}$ is given by

$$T_{rs} = b \sum_{j=1}^{v} (\bar{x}_{(r)\,0j} - \bar{x}_{r\,(00)})\,(\bar{x}_{(s)\,0j} - \bar{x}_{(s)\,00})$$

and other symbols are similarly defined. Normal equations are given by

$$A'Y = A'A\,\Theta$$

$$\begin{bmatrix} m' \\ b_1' \\ \vdots \\ b_b' \\ t_1' \\ \vdots \\ t_v' \\ c_1' \\ \vdots \\ c_p' \end{bmatrix} Y = \begin{bmatrix} m'm & m'b_1 & \dots & m'b_b & m't_1 & \dots & m't_v & m'c_1 & \dots & m'c_p \\ b_1'm & b_1'b_1 & \dots & b_1'b_b & b_1't_1 & \dots & b_1't_v & b_1'c_1 & \dots & b_1'c_p \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_b'm & b_b'b_1 & \dots & b_b'b_b & b_b't_1 & \dots & b_b't_v & b_b'c_1 & \dots & b_b'c_p \\ t_1'm & t_1'b_1 & \dots & t_1'b_b & t_1't_1 & \dots & t_1't_v & t_1'c_1 & \dots & t_1'c_p \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ t_v'm & t_v'b_1 & \dots & t_v'b_b & t_v't_1 & \dots & t_v't_v & t_v'c_1 & \dots & t_v'c_p \\ c_1'm & c_1'b_1 & \dots & c_1'b_b & c_1't_1 & \dots & c_1't_v & c_1'c_1 & \dots & c_1'c_p \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_p'm & c_p'b_1 & \dots & c_p'b_b & c_p't_1 & \dots & c_p't_v & c_p'c_1 & \dots & c_p'c_p \end{bmatrix}$$

$$\times (\hat{\mu}\ \ \hat{\alpha}_1\ \dots\ \hat{\alpha}_b\ \ \hat{\tau}_1\ \dots\ \hat{\tau}_v\ \ \hat{\xi}_1\ \ \hat{\xi}_p)'$$

Also the elements of $A'A$ are obtainable from the table below :

	m	b_i	t_j	c_r
m'	bv	v	b	0
b_i'	v	$\delta_{ii'}\,v$	1	$v\,(\bar{x}_{(s)\,i0} - \bar{x}_{(s)\,00})$
t_j'	b	1	$\delta_{jj'}\,b$	$b\,(\bar{x}_{(s)\,0j} - \bar{x}_{(s)\,00})$
c_r'	0	$v\,(\bar{x}_{(s)\,i0} - \bar{x}_{(s)\,00})$	$b\,(\bar{x}_{(s)\,0j} - \bar{x}_{(s)\,00})$	$\sum (x_{r\,(ij)} - \bar{x}_{r\,(00)})\,(\bar{x}_{s\,(ij)} - \bar{x}_{(s)\,00})$

$\rho\,(v_\alpha)$ = Rank of the estimation space

$= (b + v - 1)$ for RBD

Since p Concomittant variables are independent, total parameters are $(b + v + p - 1)$.

$\rho\,(v_\alpha) = (b + v + p - 1)$, because p concomittant variables are independent among themselves as well as independent of block and treatment effects. Hence, the normal equations can be written as

$$Y_{i0} = v\,\hat{\mu} + v\,\hat{\alpha}_i + \sum_{j=1}^{v} \hat{\tau}_j + v \sum_{s=1}^{p} \xi_s\,(\bar{x}_{(s)\,i0} - \bar{x}_{(s)\,00});\ i = 1, 2, ..., b \qquad (6.59)$$

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$$Y_{0j} = b\hat{\mu} + b\hat{\tau}_j + \sum_{i=1}^{b} \hat{\alpha}_i + b \sum_{s=1}^{p} \xi_s (\bar{x}_{(s)\,0j} - \bar{x}_{(s)\,00});\ j = 1, 2, \ldots, v \quad (6.60)$$

$$\sum_{i,j} y_{ij} (\bar{x}_{s\,(ij)} - \bar{x}_{(s)\,00}) = v \sum_{i=1}^{b} [\bar{x}_{(s)\,i0} - \bar{x}_{(s)\,00}]\,\hat{\alpha}_i + b \sum_{j=1}^{v} [\bar{x}_{(s)\,0j} - \bar{x}_{(s)\,00}]\,\hat{\tau}_j + \sum_{r,i,j} (x_{r\,(ij)} - \bar{x}_{(r)\,00})\,(x_{s\,(ij)} - \bar{x}_{(s)\,00})\,\hat{\xi}_{jr} \quad (6.61)$$

As the rank of the estimation space is $b + v - 1 + p$ (because of p independent concomittant variables introduced), we impose two restrictions, viz.,

$$\sum_{i=1}^{b} \hat{\alpha}_i = 0 \text{ and } \sum_{j=1}^{v} \hat{\tau}_j = 0$$

to get particular solutions of Normal equations. This gives

$$\hat{\mu} = Y_{00}/bv = \bar{Y}_{00} \quad (6.62)$$

and Normal equation (6.59) gives

$$v(\hat{\mu} + \hat{\alpha}_i) = Y_{i0} - v \sum_{s=1}^{p} \xi_s (\bar{x}_{(s)\,i0} - \bar{x}_{(s)\,00})$$

or

$$\hat{\alpha}_i = \bar{Y}_{i0} - \sum_{s=1}^{p} \xi_s (\bar{x}_{(s)\,i0} - \bar{x}_{(s)\,00}) - \bar{Y}_{00} \quad (6.63)$$

Similarly from (6.60),

$$\hat{\tau}_j = \bar{Y}_{0j} - \sum_{s=1}^{p} \xi_s (\bar{x}_{(s)\,0j} - \bar{x}_{(s)\,00}) - \bar{Y}_{00} \quad (6.64)$$

Putting (6.62), (6.63) and (6.64) in (6.61), we have

$$\sum_{i,j} y_{ij} (x_{s\,(ij)} - \bar{x}_{(s)\,00})$$

$$= v \sum_{i=1}^{b} [(\bar{x}_{(s)\,i0} - \bar{x}_{(s)\,00})\,\{(\bar{Y}_{i0} - \bar{Y}_{00}) - \Sigma\, \xi_r (\bar{x}_{(r)\,i0} - \bar{x}_{(r)\,00})\}]$$

$$+ b \sum_{j=1}^{v} [(\bar{x}_{(s)\,0j} - \bar{x}_{(s)\,00})\,\{(\bar{Y}_{0j} - \bar{Y}_{00}) - \Sigma\, \xi_r (\bar{x}_{(r)\,0j} - \bar{x}_{(r)\,00})\}]$$

$$+ \sum_{r,i,j} (x_{r\,(ij)} - \bar{x}_{(r)\,00})\,(x_{s\,(ij)} - \bar{x}_{(s)\,00})\,\hat{\xi}_{jr} \quad (6.65)$$

$$= B_{sy} - \sum_{r=1}^{p} B_{rs}\hat{\xi}_{jr} + T_{sy} - \sum_{r=1}^{p} T_{rs}\xi_{jr} + \sum_{r=1}^{p} G_{rs}\xi_{jr} = G_{sy}$$

$$\Rightarrow \quad G_{sy} - B_{sy} - T_{sy} = \sum_{r=1}^{p} \xi_r (G_{rs} - B_{rs} - T_{rs})$$

or

$$E_{sy} = \sum_{r=1}^{p} \xi_r E_{rs};\ s = 1, 2, \ldots, p$$

$$E_{sy} = (\xi_1\ \xi_2 \ldots \xi_p) \begin{pmatrix} E_{1s} \\ E_{2s} \\ \vdots \\ E_{ps} \end{pmatrix}$$

$$\Sigma\, \xi_s E_{rs} = E_{ry}$$

$$(E_{rs})^{-1} \begin{pmatrix} E_{1y} \\ E_{2y} \\ \vdots \\ E_{py} \end{pmatrix} = (\xi_1\ \xi_2 \ldots \xi_p)'$$

provided $(E_{rs})^{-1}$ exists. Now it remains to be shown that $(E_{rs})_{p \times p}$ is a non-singular matrix, i.e., to show that $\rho\,(E_{rs}) = p$.

Also

$$\zeta_r = [x_{r\,(ij)} - \bar{x}_{r\,(i0)} - \bar{x}_{r\,(0j)} + \bar{x}_{r\,(00)}];\ i = 1, 2, \ldots, b;\ j = 1, 2, \ldots, v$$

$$\zeta_r' \zeta_s = \sum_{i,j} [x_{r\,(ij)} - \bar{x}_{r\,(i0)} - \bar{x}_{r\,(0j)} - \bar{x}_{r\,(00)}]\,[\bar{x}_{r\,(ij)} - \bar{x}_{r\,(i0)} - \bar{x}_{r\,0j)} + \bar{x}_{r\,(00)}]$$

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$$\Rightarrow \qquad E_{rs} = \begin{pmatrix} \zeta_1' \\ \zeta_2' \\ \vdots \\ \zeta_p' \end{pmatrix} (\zeta_1 \quad \zeta_2 \ldots \zeta_p)$$

Now $\rho(E_{rs}) = \rho(\zeta_1, \zeta_2 \ldots, \zeta_p) = \rho(\xi_1, \xi_2 \ldots, \xi_p) = p$

$\Rightarrow$ E_{rs} is a non-singular matrix

S.S due to estimates is $\hat{\Theta}'A'Y$ with $(b + v + p - 1)$ d.f. From (6.62), (6.63) and (6.64), we have

$$\hat{\mu} = Y_{00}/bv = \bar{Y}_{00}$$

$$\hat{\alpha}_i = (\bar{Y}_{i0} - \bar{Y}_{00}) - \sum_{s=1}^{p} \xi_s (\bar{x}_{(s)\,i0} - \bar{x}_{(s)\,00})$$

and

$$\hat{\tau}_j = (\bar{Y}_{0j} - \bar{Y}_{00}) - \sum_{s=1}^{p} \xi_s (\bar{x}_{(s)\,0j} - \bar{x}_{(s)\,00})$$

Also $\quad C_r'Y = \sum_{ij} (x_{r\,(ij)} - \bar{x}_{r\,(00)})(Y_{ij} - \bar{Y}_{00})$

$\therefore$ S.S due to estimates

$$= bv(\bar{Y}_{00})^2 + v \sum_{i=1}^{b} [\bar{Y}_{i0} \{(\bar{Y}_{i0} - \bar{Y}_{00}) - \sum_{s=1}^{p} \xi_s (\bar{x}_{(s)\,i0} - \bar{x}_{(s)\,00})\}]$$

$$+ b \sum_{j=1}^{v} [\bar{Y}_{0j} \{(\bar{Y}_{0j} - \bar{Y}_{00}) - \sum_{s=1}^{p} \xi_s (\bar{x}_{(s)\,0j} - \bar{x}_{(s)\,00})\}] + \sum_{r=1}^{p} \xi_r G_{rY}$$

with $(b + v + p - 1)$ d.f. Now

$$v \sum_i \bar{Y}_{i0} (\bar{Y}_{i0} - \bar{Y}_{00}) = v \sum_i (\bar{Y}_{i0} - \bar{Y}_{00})(\bar{Y}_{i0} - \bar{Y}_{00}) = B_{YY}$$

and

$$b \sum_j \bar{Y}_{0j} (\bar{Y}_{0j} - \bar{Y}_{00}) = b \sum_j (\bar{Y}_{0j} - \bar{Y}_{00})(\bar{Y}_{0j} - \bar{Y}_{00}) = T_{YY}$$

$\therefore$ S.S due to estimates

$$= bv(\bar{Y}_{00})^2 + B_{yy} - \Sigma \hat{\xi}_s B_{sY} + T_{YY} - \Sigma \hat{\xi}_s T_{sY} + \Sigma \hat{\xi}_s G_{sY}$$

$$= bv(\bar{Y}_{00})^2 + \Sigma \hat{\xi}_s (G_{sY} - B_{sY} - T_{sY}) + (T_{YY} + B_{YY})$$

$$= bv(\bar{Y}_{00})^2 + \Sigma \hat{\xi}_s E_{sY} + (G_{YY} - E_{YY})$$

with $(b + v + p - 1)$ d.f. (6.66)

S.S due to error $= Y'Y - \hat{\Theta}'A'Y$

$$= \Sigma y_{ij}^2 - bv(\bar{Y}_{00})^2 - \Sigma \hat{\xi}_s E_{sY} - G_{YY} + E_{YY}$$

with $(bv - v - b - p + 1) = (b - 1)(v - 1) - p$ d.f.

$\therefore$ S.S due to error $= G_{YY} - \Sigma \hat{\xi}_s E_{sY} - G_{YY} + E_{YY}$

$$= E_{YY} - \Sigma \hat{\xi}_s E_{sY}$$

with $(b - 1)(v - 1) - p$ d.f.

Testing of Hypothesis

$$H_0 : (\xi_1 = \xi_2 = \ldots = \xi_p = 0)$$

S.S due to hypothesis H_0 = S.S due to error under H_0 − S.S due to error (unconditional)

$$= E_{YY} - E_{YY} + \Sigma \hat{\xi}_s E_{sY}$$

$$= \Sigma \hat{\xi}_s E_{sY}$$

with $(b - 1)(v - 1) - [(b - 1)(v - 1) - p] = p$ d.f.

The appropriate test for H_0 is

$$= \frac{\Sigma \hat{\xi}_s E_{sY}/p}{E_{YY} - \Sigma \hat{\xi}_s E_{sY}/[(b-1)(v-1) - p]} \qquad (6.67)$$

which is distributed as central F with p and $[(b - 1)(v - 1) - p]$ d.f.

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To test H_0' : $(\tau_1 = \tau_2 = \ldots = \tau_v = T)$

Model : $$E(Y) = m\mu + \sum_{i=1}^{b} b_i \alpha_i + m\tau + \sum_{r=1}^{p} c_r \xi_r$$

$$= m(\mu + \tau) + \sum_{i=1}^{b} b_i \alpha_i + \sum_{r=1}^{p} c_r \xi_r$$

$$\therefore \quad \hat{\alpha}_i = (\bar{Y}_{i0} - \bar{Y}_{00}) - \sum_{s=1}^{p} \xi_s^* (\bar{x}_{(s)\,i0} - \bar{x}_{(s)\,00})$$

where ξ_s^* is given by $\Sigma E_{rs} \xi_s^* = E_{ry}$; $r = 1, 2, \ldots, p$

$$\sum_{s=1}^{p} (G_{rs} - B_{rs}) \xi_s^* = (G_{ry} - B_{ry}) \tag{6.68}$$

and $$\sum_{s=1}^{p} (T_{rs} + E_{rs}) \xi_s^* = (T_{ry} + E_{ry}) \tag{6.69}$$

ξ_s^* can be obtained from the solution of the above system of equations.

S.S due to estimates under H_0'

$$= bv(\bar{Y}_{00})^2 + v \sum_{i=1}^{b} [\bar{Y}_{i0} \{(\bar{Y}_{i0} - \bar{Y}_{00}) - \sum_{s=1}^{p} \xi_s^* (\bar{x}_{(s)\,i0} - \bar{x}_{(s)\,00})\}]$$

$$+ \sum_{s=1}^{p} \xi_s^* G_{sY} \text{ with } (b + v + p - 1) - (v - 1) \text{ d.f.}$$

S.S due to error under H_0'

$= Y'Y$ – S.S due to estimates under H_0'

$$= [Y'Y - bv(\bar{Y}_{00})^2] - B_{YY} + \Sigma \xi_s^* B_{sY} - \Sigma \xi_s^* G_{sY}$$

$$= G_{YY} - B_{YY} + \Sigma \xi_s^* (G_{sY} - B_{sY})$$

$$= T_{YY} + E_{YY} - \Sigma \xi_s^* (T_{sY} + E_{sY})$$

with $\{bv - (b + p)\} = b(v - 1) - p$ d.f.

S.S due to hypothesis under H_0' = S.S due to error under H_0' – S.S due to error (unconditional)

$$= T_{YY} + E_{YY} - \Sigma \xi_s^* (T_{sY} + E_{sY}) - E_{YY} + \Sigma \xi_s^* (E_{sY})$$

$$= T_{YY} - \Sigma \xi_s^* (T_{sY} + E_{sY}) + \Sigma \xi_s^* (E_{sY})$$

with $(bv - b - p) - \{(b - 1)(v - 1) - p\} = v - 1$ d.f.

The appropriate test given by

$$F = \frac{T_{YY} - \Sigma \xi_s^* (T_{sY} + E_{sY}) + \Sigma \xi_s^* (E_{sY})/(v - 1)}{E_{YY} - \Sigma \xi_s^* (E_{sY})/[(b - 1)(v - 1) - p]}$$

is distributed as Snedecor's F with $(v - 1)$ and $(b - 1)(v - 1) - p$ d.f.

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PROBLEMS AND SOLUTIONS

Problem 1 : If X and Y are bivariate normally distributed variables, then the regression of Y on X (and X on Y) is linear and $f(x, y)$ is given by

$$\frac{e^{\frac{-1}{2(1-\rho^2)^{1/2}}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]}}{2\pi\,\sigma_x\,\sigma_y\,\sqrt{1-\rho^2}}$$

$$-\infty < x < \infty,\ -\infty < y < \infty$$

The conditional density function of $Y \mid X = x$ is given by

$$f(y \mid x) = \frac{f(x, y)}{f(x)},\ f(x) > 0$$

$$= \frac{e^{-\frac{1}{2(1-\rho^2)(1/2)}\left[(1-\rho^2)\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 + \rho^2\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right]}}{2\pi\,\sigma x\,\sigma y\,\sqrt{1-\rho^2}}$$

$$\frac{e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}{\sqrt{2\pi}\,\sigma_x}$$

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$$= \frac{\frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} e^{-\frac{1}{2}(1-\rho^2)\left[\left(\frac{y-\mu_y}{\sigma_y}\right)-\rho\,\sigma_y\left(\frac{x-\mu_x}{\sigma_x\,\sigma_y}\right)\right]^2}}{2\pi\,\sigma_x\,\sigma_y\sqrt{1-\rho^2}\, e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}$$

$$= \frac{1}{\sqrt{2\pi\,(1-\rho^2)}\,\sigma_y} e^{-\frac{1}{2}\left[\frac{(1-\rho^2)}{\sigma_y^2}\right]\left[(y-\mu_y)-\rho\frac{\sigma_y}{\sigma_x}(x-\mu_x)\right]^2}$$

The conditional mean $Y \mid X = x$ is, therefore

$$E(Y \mid X = x) = \mu_y + \rho\frac{\sigma_y}{\sigma_x}(x - \mu_x)$$

which is linear in x. This proves that under the condition, the regression of Y on X is linear. Precisely, by a similar arguements, the regression of X on Y is given by

$$E(X \mid Y = y) = \mu_x + \rho\frac{\sigma_x}{\sigma_y}(y - \mu_y)$$

is also linear.

Remarks : The converse of the result does not hold good. However, if X_1 and X_2 are bivariate normal

$$Q = \sum_{j=1}^{2}\sum_{k=1}^{2} a_{jk} X_j X_k + \sum_{j=1}^{2} b_j X_j$$

has a constant regression on $L = X_1 + X_2$ subject to the condition

$$B_1 = \sum_{j=1}^{2} a_{ij} \neq 0,\quad B_2 = \sum_{j=1}^{2}\sum_{k=1}^{2} a_{jk} = 0,$$

and

$$B_3 = \sum_{j=1}^{2} b_j = 0$$

The constant regression implies Homoscedasticity of the condition-al variance Q on L (Mathai and Pederzoli, 1977).

Problem 2 : Let (X, Y) have the joint p.d.f. given by

$$f(x, y) = 1 \text{ if } x > |y|;\ 0 < x < 1$$

$$= 0 \text{ otherwise}$$

show that the regression of Y on X is linear whereas that between X on Y is non-linear (Biswas, 1991).

We have $f(x \mid y) = 1/(1 - |y|),\ 0 < y < 1$

because, $f(x \mid y) = \frac{f(x, y)}{f(y)} = \frac{1}{(1-y)};\ 0 < y < 1$

Again, $f(y \mid x) = \frac{f(x, y)}{f(x)} = \frac{1}{2x};\ |y| < x$

$$\Rightarrow \quad E(Y \mid x) = \int_{-x}^{x} \frac{1}{2x} y\,dy = 0$$

i.e. Regression of Y on X is the line $y = 0$, the X-axis, whereas

$$E(X \mid y) = \int_{0}^{1} \frac{1}{1-y} x\,dx$$

$$= \frac{1}{2(1-y)};\ 0 < y < 1$$

and is equal to $\int_{0}^{1} \frac{1}{1+y} x\,dx$

$$= \frac{1}{2(1+y)};\ -1 < y \leq 0$$

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$$\Rightarrow \qquad E(x \mid y) = \frac{1}{2(1 + \mid y \mid)}$$

which is non-linear.

Problem 3 : Let $E(Y \mid X = x) = \alpha + \beta (x_i - \bar{x})$

but $$\text{Var}(y_i) = \frac{1}{w_i}\sigma_i^2 \quad \text{cov}(y_i, y_j)$$

$$= 0$$

(non-homoscedastic variance as it arises in consumption and expenditure data given by different social or economic strata). Then obtain the S.S due to regression and total S.S.

The weighted S.S

$$Q = \sum_{i=1}^{k} \sum_{j=1}^{n_i} w_i \{y_{ij} - \alpha - \beta (x_i - \bar{x})\}^2$$

Minimising Q w.r.t. α and β we have

$$\hat{\alpha} = \frac{\sum_{i=1}^{k} n_i w_i y_i}{\sum_{i=1}^{k} n_i w_i}$$

$$\hat{\beta} = \frac{\sum_{i=1}^{k} n_i w_i \bar{y}_i (x_i - \bar{x})}{\sum_{i=1}^{k} n_i w_i (x_i - \bar{x})^2}$$

where $$y_i = \sum_{j=1}^{n_i} y_{ij};\ \bar{y}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} y_{ij} \text{ and } \bar{x} = \frac{\sum n_i w_i x_i}{\sum n_i w_i}$$

S.S due to regression

$$= \hat{\beta} \sum_i n_i w_i \bar{y}_i (x_i - \bar{x})$$

$$= \hat{\beta}^2 \sum_i n_i w_i (x_i - \bar{x})^2$$

Total S.S $$= \sum_{i=1}^{k} \sum_{j=1}^{n_i} w_i y_{ij}^2 - \frac{(\sum n_i w_i \bar{y}_i)^2}{\sum n_i w_i}$$

$$R^2 = \frac{\hat{\beta}^2 \sum_i n_i w_i (x_i - \bar{x})^2}{\sum_{i=1}^{k} \sum_{j=1}^{n_i} w_i y_{ij}^2 - \frac{(\sum n_i w_i \bar{y}_i)^2}{\sum n_i w_i}}$$

which is the square of Multiple Correlation coefficient.

Problem 4 : To test H_0 :

(*i*) $\beta_i = b_i,\ \forall\ i = 1, 2, \ldots, p$

(*ii*) $\beta_i = 0$ against $\beta_i \neq 0,\ i = 1, 2, \ldots, p$

(*iii*) $\beta_1 = \beta_2 = \ldots = \beta_p = \beta$

(*iv*) $\beta_1 + \beta_2 + \ldots + \beta_p = C$, a given constant

(*v*) $\beta_i = \beta_i^0$ for some $i = 1, 2, \ldots, p$

(*vi*) A few coefficients in the multiple regression model are zero

(*vii*) Equality of two regression lines.

Theoretical Background (An Alternative Treatment)

The background for the testing of above hypotheses are as follows:

Let $E(Y) = \alpha + \beta_1 (x_1 - \bar{x}_1) + \beta_2 (x_2 - \bar{x}_2) + \ldots + \beta_p (x_p - \bar{x}_p)$

be the multiple regression line of Y on $X_1, X_2, \ldots, X_p$. We can write the l^{th} observation ($l = 1, 2, \ldots, n$)

$$y_l = \alpha + \beta_1 (x_1 - \bar{x}_1) + \beta_2 (x_2 - \bar{x}_2) + \ldots + \beta_p (x_p - \bar{x}_p) + \epsilon_l$$

where $E(\epsilon_l) = 0,\ \text{Var}(\epsilon_l) = \sigma^2,\ \text{Cov}(\epsilon_l, \epsilon_l) = 0$

Minimising residual S.S w.r.t. $\alpha, \beta_1, \beta_2, \ldots, \beta_p$, we get the normal equations,

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$$n\bar{Y} = n\alpha \Rightarrow \bar{Y} = \alpha$$

$$\sum_l (x_{1l} - \bar{x}_1)\, y_l = \hat{\alpha} \sum (x_{1l} - \bar{x}_1) + \hat{\beta}_1 \sum (x_{1l} - \bar{x}_1)^2 + \ldots$$

$$\ldots + \hat{\beta}_p \sum (x_{1l} - \bar{x}_1)\,(x_{pl} - \bar{x}_p)$$

$$\sum_l (x_{2l} - \bar{x}_2)\, y_l = \hat{\alpha} \sum (x_{2l} - \bar{x}_2) + \hat{\beta}_1 \sum (x_{1l} - \bar{x}_1)\,(x_{2l} - \bar{x}_2) + \ldots$$

$$+ \hat{\beta}_p \sum (x_{2l} - \bar{x}_2)\,(x_{pl} - \bar{x}_p)$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\sum_l (x_{pl} - \bar{x}_p)\, y_l = \hat{\alpha} \sum (x_{pl} - \bar{x}_p) + \hat{\beta}_1 \sum (x_{1l} - \bar{x}_1)\,(x_{pl} - \bar{x}_p)$$

$$+ \ldots + \hat{\beta}_p \sum (x_{pl} - \bar{x}_p)^2$$

$$\Rightarrow \quad \sum (x_{il} - \bar{x}_i)\, y_l = \hat{\beta}_1 \sum (x_{il} - \bar{x}_1)\,(x_{il} - \bar{x}_i) + \ldots$$

$$\hat{\beta}_p \sum (x_{pl} - \bar{x}_p)\,(x_{il} - \bar{x}_i),\ i = 1, 2, \ldots, p$$

$[\text{as} \sum (x_{il} - \bar{x}_i) = 0]$

Putting $\sum (x_{il} - \bar{x}_i)\, y_l = g_i$, $\sum (x_{il} - \bar{x}_i)\,(x_{jl} - \bar{x}_j) = s_{ij}$, we get

$$\begin{bmatrix} g_1 \\ \vdots \\ g_i \\ \vdots \\ g_p \end{bmatrix}_{p \times 1} = \begin{bmatrix} s_{11} & s_{12} & \ldots & s_{1p} \\ \vdots & \vdots & \vdots & \vdots \\ s_{i1} & s_{i2} & \ldots & s_{ip} \\ \vdots & \vdots & \vdots & \vdots \\ s_{p1} & s_{p2} & \ldots & s_{pp} \end{bmatrix}_{p \times p} \begin{bmatrix} \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_i \\ \vdots \\ \hat{\beta}_p \end{bmatrix}_{p \times 1}$$

$$\Rightarrow \quad (S_{ij})^{-1} \begin{bmatrix} g_1 \\ \vdots \\ g_i \\ \vdots \\ g_p \end{bmatrix} = \begin{bmatrix} \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_i \\ \vdots \\ \hat{\beta}_p \end{bmatrix}$$

or $$(S_{ij}) \begin{bmatrix} g_1 \\ \vdots \\ g_i \\ \vdots \\ g_p \end{bmatrix} = \begin{bmatrix} \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_i \\ \vdots \\ \hat{\beta}_p \end{bmatrix}$$

where $(S^{ij}) = (S_{ij})^{-1}$

Also it is easily shown that

$$E(g_i) = \sum_{j=1}^{p} \beta_j s_{ji}$$

$$\text{Cov}(g_i, g_j) = \sigma^2 \sum (x_{il} - \bar{x}_i)\,(x_{jl} - \bar{x}_j)$$

$$= \sigma^2 s_{ij}$$

$$\Rightarrow \quad \text{Var}(g_i) = \sigma^2 s_{ii}$$

$$E(\hat{\beta}_i) = \sum_{k=1}^{p} s^{ik} E(g_k)$$

$$= \sum_{k=1}^{p} s^{ik} \sum_{j=1}^{p} \beta_j s_{jk}$$

$$= \sum_{j=1}^{p} \beta_j \sum_{k=1}^{p} s^{ik} s_{jk}$$

$$= \sum_{j=1}^{p} \left\{ \beta_j \sum_{k=1}^{p} s^{ik} s_{kj} \right\}$$

(the matrix being symmetric)

$$= \beta_i \ (\text{since } (s^{ij}) = (s_{ij})^{-1}) \qquad (1)$$

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Hence, $\hat{\beta}_i$ is unbiased for β_i

$$\text{Cov}(\hat{\beta}_i, \hat{\beta}_j) = \sigma^2 \sum_{l=1}^{p} \sum_{k=1}^{p} s^{il} s^{jk} s_{lk}$$

$$= \sigma^2 \sum_{k=1}^{p} s^{jk} \sum_{l=1}^{p} s^{il} s_{lk}$$

$$= \sigma^2 s^{ij}$$

$$= s^{ij} \sigma^2 \qquad (2)$$

[because $\sum_{l} s^{il} s_{lk} = 1$ for $k = i$

$= 0$ otherwise]

$$\text{Var}(\hat{\beta}_i) = s^{ii} \sigma^2$$

Thus, the dispersion matrix of $\hat{\beta}_i = S^{ij} \sigma^2$ (3)

Sampling Distribution of $\hat{\beta}$

We have, $\hat{\beta}_i = s^{i1} g_1 + s^{i2} g_2 + \ldots + s^{ip} g_p$

If $\epsilon_l \sim N(0, \sigma^2)$, $\hat{\beta}_i$ is a linear combination of the normally distributed variables.

$$E(\hat{\beta}_i) = \beta_i, \quad \text{Var}(\hat{\beta}_i) = s^{ii} \sigma^2 \quad \text{and} \quad \text{Cov}(\hat{\beta}_i, \hat{\beta}_j) = s^{ij} \sigma^2$$

Joint distribution of $\hat{\beta}_i$'s is multivariate Normal of the form

$$-\frac{1}{2\sigma^2} (\hat{\beta} - \beta)' (s^{ij})^{-1} (\hat{\beta} - \beta)$$

where $C = (2\pi)^{-p/2} |s_{ij}|^{-1/2} \sigma^{-p}$; hence

$$(\hat{\beta} - \beta)' (s^{ij})^{-1} (\hat{\beta} - \beta) = \sum_{i,j=1}^{p} (\hat{\beta}_i - \beta_i)' (s^{ij})^{-1} (\hat{\beta}_j - \beta_j) \qquad (4)$$

is $\chi^2 \sigma^2$ with p.d.f.

An Unbiased Estimate of σ^2

Consider the multiple regression of Y on $X_1, X_2, \ldots, X_p$. Let y_l and Y_l be the observed and expected values (on the basis of the linear regression) corresponding to the l^{th} observation. $(l = 1, 2, \ldots, n)$. Then

$$\sum_{l=1}^{n} (y_l - Y_l)^2 = \sum_{l=1}^{n} [y_l - \bar{y} - \hat{\beta}_1 (x_{1l} - \bar{x}_1) - \ldots - \hat{\beta}_p (x_{pl} - \bar{x}_p)]^2$$

$$= \sum_{l=1}^{n} (y_l - \bar{y})^2 - 2 \sum_{j=1}^{p} \hat{\beta}_j \sum_{1l-1}^{n} (y_l - \bar{y}) (x_{jl} - \bar{x}_j)$$

$$+ \sum_{i,j=1}^{p} \hat{\beta}_i \hat{\beta}_j \sum_{l=1}^{n} (x_{il} - \bar{x}_i) (x_{jl} - \bar{x}_j)$$

$$= \sum_{l=1}^{n} (y_l - \bar{y})^2 - 2 \sum_{j=1}^{p} \hat{\beta}_j \sum_{l=1}^{n} y_l (x_{jl} - \bar{x}_j)$$

$$+ \sum_{i,j=1}^{p} \hat{\beta}_i \hat{\beta}_j \sum_{l=1}^{n} (x_{il} - \bar{x}_i) (x_{jl} - \bar{x}_j)$$

[since $\sum_{l} \bar{y} (x_{jl} - \bar{x}_j) = 0$]

$$= \sum_{l=1}^{n} (y_l - \bar{y})^2 - 2 \sum_{j=1}^{p} \hat{\beta}_j g_j$$

$$+ \sum_{i,j=1}^{p} \hat{\beta}_i \hat{\beta}_j \sum_{l=1}^{n} (x_{il} - \bar{x}_i) (x_{jl} - \bar{x}_j)$$

$$= \sum_{l=1}^{n} (y_l - \bar{y})^2 - 2 \sum_{j=1}^{p} \hat{\beta}_j g_j + \sum_{i} \hat{\beta}_i (x_{il} - \bar{x}_i) (x_{jl} - \bar{x}_j) \sum_{j} \hat{\beta}_j$$

[Since we have from normal equations

$$\hat{\beta}_i \sum_{1} (x_{il} - \bar{x}_i) (x_{jl} - \bar{x}_j) = \sum_{l} (x_{jl} - \bar{x}_j) y_l = g_j]$$

$$= \sum_{l=1}^{n} (y_l - \bar{y})^2 - 2 \sum_{j=1}^{p} \hat{\beta}_j g_j + \sum_{j=1}^{p} \hat{\beta}_j g_j$$

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$$= \sum_{l=1}^{n} (y_l - \bar{y})^2 - \sum_{j=1}^{p} \hat{\beta}_j g_j$$

$$= \sum_{l=1}^{n} y_l^2 - n\bar{y}^2 - \sum_{j=1}^{p} \hat{\beta}_j g_j \tag{5}$$

$\Sigma \hat{\beta}_j g_j \equiv$ S.S due to estimates of $\hat{\beta}_1 \hat{\beta}_2 \ldots \hat{\beta}_p$ (6)

$n\bar{y}^2 \equiv$ S.S due to estimate of α (7)

$\Sigma y_l^2 \equiv$ Total S.S

Hence, R.H.S. of (5) is the S.S due to errors (8)

Next we have

$$y_l = \alpha + \beta_1 (x_{1l} - \bar{x}_1) + \ldots + \beta_p (x_{pl} - \bar{x}_p) + \epsilon_l$$

$$Y_l = \bar{y} + \hat{\beta}_1 (x_{1l} - \bar{x}_1) + \ldots + \hat{\beta}_p (x_{pl} - \bar{x}_p)$$

$$y_l - Y_l = \alpha - \bar{y} + (\beta_1 - \hat{\beta}_1)(x_{1l} - \bar{x}_1) + \ldots + (\beta_p - \hat{\beta}_p)(x_{pl} - \bar{x}_p) + \epsilon_l$$

Again $\bar{y} = \alpha + \bar{\epsilon} \Rightarrow \alpha - \bar{y} = -\bar{\epsilon}$

$$\therefore \quad y_l - Y_l = -(\hat{\beta}_1 - \beta_1)(x_{1l} - \bar{x}_1) - \ldots - (\hat{\beta}_p - \beta_p)(x_{pl} - \bar{x}_p) + \epsilon_l - \bar{\epsilon}$$

$$\sum_l (y_l - Y_l)^2 = \sum_l [(\epsilon_l - \bar{\epsilon}) - \sum_i (\hat{\beta}_i - \beta_i)(x_{il} - \bar{x}_i)]^2$$

$$= \sum_l (\epsilon_l - \bar{\epsilon})^2 - \sum_{i,j} (\hat{\beta}_i - \beta_i)(x_{il} - \bar{x}_i)(\hat{\beta}_j - \beta_j)(x_{jl} - \bar{x}_j)$$

$$= \sum_l (\epsilon_l - \bar{\epsilon})^2 - \sum_{i,j} (\hat{\beta}_i - \beta_i)(\hat{\beta}_j - \beta_j) \sum_l (x_{il} - \bar{x}_i)(x_{jl} - \bar{x}_j)$$

$$= \sum_l (\epsilon_l - \bar{\epsilon})^2 - \sum_{i,j} (\hat{\beta}_i - \beta_i)(\hat{\beta}_j - \beta_j) s_{ij} \tag{9}$$

But $E\left[\sum_l (\epsilon_l - \bar{\epsilon})^2\right] = (n-1)\sigma^2$

and $E\left[\sum_{i,j} (\hat{\beta}_i - \beta_i)(\hat{\beta}_j - \beta_j) s_{ij}\right] = E\{\text{Cov}(b_i, b_j)\}$

$$= s^{ij} \sigma^2$$

$$= \sigma^2 \sum_{i,j} s^{ij} s_{ij}$$

$$= p\sigma^2$$

Thus (10)

$$E[\Sigma (y_l - Y_l)^2] = E[\Sigma (y_l - \bar{y})^2 - \Sigma b_i g_i]$$

$$= E[\sum_l (\epsilon_l - \bar{\epsilon})^2] - E\left[\sum_{i,j} (\hat{\beta}_i - \beta_i)(\hat{\beta}_j - \beta_j) s_{ij}\right]$$

$$= (n-1)\sigma^2 - p\sigma^2$$

$$= (n-p-1)\sigma^2 \tag{11}$$

i.e. $$\frac{\Sigma (y_l - y)^2 - \Sigma b_i g_i}{(n-p-1)} \tag{12}$$

is an unbiased estimate of σ^2. It is also to be noted that we did not make use of the assumption of normality of the dependent variable for the purpose of estimation.

Testing of Linear Hypothesis

$$\Sigma (y_l - Y_l)^2 = \sum_{l=1}^{n} (\epsilon_l - \bar{\epsilon})^2 - \sum_{i,j=1}^{p} (\hat{\beta}_i - \beta_i)(\hat{\beta}_j - \beta_j) s_{ij}$$

Now $\sum_{l=1}^{n} (\epsilon_l - \bar{\epsilon})^2 \sim \chi^2 \sigma^2$ with $(n-1)$ d.f. (13)

$\sum_{i,j=1}^{p} (\hat{\beta}_i - \beta_i)(\hat{\beta}_j - \beta_j) s_{ij} \sim \chi^2 \sigma^2$ with p d.f. (14)

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Also $(\epsilon_l - \bar{\epsilon})$ and $(b_i - \beta_i)$ are independent.

Thus, $\sum_{l=1}^{n} (y_l - Y_l)^2 = \sum_{l=1}^{n} (y_l - \bar{y})^2 - \sum_{i=1}^{p} \hat{\beta}_i g_i$

$$= \sum_{l=1}^{n} (\epsilon_l - \bar{\epsilon})^2 - \sum_{i,j=1}^{p} (\hat{\beta}_i - \beta_i)(\hat{\beta}_j - \beta_j) s_{ij}$$

$$\sim \chi^2 \sigma^2 \text{ with } (n-1-p) \text{ d.f.} \qquad (15)$$

and is independent of any hypothesis concerning the parameters.

Specific Hypothesis

(i) $H_0 : \beta_i = b_i \ (i = 1, 2, ..., p)$

Under H_0, $\sum_{i,j=1}^{p} (b_i - \beta_i)(b_j - \beta_j) s_{ij} \sim \chi^2 \sigma^2$ with p d.f.

Again $\sum_{l=1}^{n} (y_l - \bar{y})^2 - \sum_{i=1}^{p} b_i g_i \sim \chi^2 \sigma^2$ with $(n-p-1)$ d.f. is the unrestricted error sum of squares

$$F = \frac{\sum_{i,j=1}^{p} (b_i - \beta_i)(b_j - \beta_j)/p}{\sum_{l=1}^{n} (y_l - \bar{y})^2 - \sum_{i=1}^{p} b_i g_i/(n-p-1)}$$

is Snedecor's F with p and $(n-p-1)$ d.f. If the calculated value of $F > 100\,\alpha\%$ tabular value of F with p and $(n-p-1)$ d.f., then H_0 is rejected. Otherwise we have no evidence to reject the same.

(ii) $H_0 : \beta_i = 0 \ (i = 1, 2, ..., p)$

Under H_0, $\sum_{i,j=1}^{p} \hat{\beta}_i \hat{\beta}_j s_{ij} = \sum_{i=1}^{p} \hat{\beta}_i g_i \sim \chi^2 \sigma^2$ with p d.f., by (14)

$\sum_{l=1}^{n} (y_l - \bar{y})^2 - \sum_{i=1}^{p} \beta_i g_i \sim \chi^2 \sigma^2$ with $(n-p-1)$ d.f., by (15)

$$F = \frac{\sum_{i=1}^{p} \hat{\beta}_i g_i/p}{\sum_{l=1}^{n} (y_l - \bar{y})^2 - \sum_{i=1}^{p} \hat{\beta}_i g_i/(n-p-1)}$$

is the appropriate test for H_0.

ANOVA TABLE OF THE TEST

Source of Variation	d.f.	Sum of Squares	Mean Square Error	F
due to $b_1, b_2, ..., b_p$	p	$\sum b_i g_i$	$\frac{\sum b_i g_i}{p} = (i)$	$\frac{(i)}{(ii)}$
Error	$n-p-1$	$\sum (y_l - \bar{y})^2 - \sum b_i g_i$	$\frac{\sum (y_l - \bar{y})^2 - \sum b_i g_i}{n-p-1} = (ii)$	
Total	$n-1$	$\sum (y_l - \bar{y})^2$		

(iii) $H_0 : \beta_1 = \beta_2 = ... = \beta_p = \beta$

We have, under H_0,

$$g_i = \sum_{l=1}^{n} (x_{il} - \bar{x}_i)\, y_l$$

and $$y_l = \alpha + \beta\,[(x_{1l} - \bar{x}_1) + ... + (x_{pl} - \bar{x}_p)] + \epsilon_l$$

$\therefore$ the normal equation for the solution of $\hat{\beta} = b$ is given by

$$\begin{bmatrix} g_1 \\ \vdots \\ g_i \\ \vdots \\ g_p \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & \dots & s_{1p} \\ \vdots & \vdots & \vdots & \vdots \\ s_{i1} & s_{i2} & \dots & s_{ip} \\ \vdots & \vdots & \vdots & \vdots \\ s_{p1} & s_{p2} & \dots & s_{pp} \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \vdots \\ \hat{\beta} \\ \vdots \\ \hat{\beta} \end{bmatrix}$$

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$$\Rightarrow \quad g_i = (s_{i1} + s_{i2} + \ldots + s_{ip})\,\hat{\beta};\ i = 1, 2, \ldots, p$$

$$\Rightarrow \quad \Sigma\, g_i = \left(\sum_{j=1}^{p} s_{ij}\right)\hat{\beta}$$

$$\Rightarrow \quad \hat{\beta} = b = \frac{\Sigma\, g_i}{\Sigma\, s_{ij}} \text{ and } \hat{\alpha} = \bar{y}$$

The sum of squares due to estimates $\hat{\alpha} = a$ and $\hat{\beta} = b$ is given by $\Sigma\, \hat{\beta}_j\, g_j + n\bar{y}^2$; from (6) and (7)

Under H_0, S.S due to estimate

$$= \hat{\beta}\ \Sigma_j\ g_j + n\bar{y}^2$$

$$= \hat{\beta} g + n\bar{y}^2, \text{ where } g = \Sigma\, g_j$$

$$= \Sigma_j\ (s_{j1} + s_{j2} + \ldots + s_{jp})\,\hat{\beta} + n\bar{y}^2$$

The error S.S under

$$H_0 = \Sigma\ (y_l - \bar{y})^2 - \hat{\beta}\ \Sigma_j\ g_j$$

$$= \sum_{l=1}^{n} y_l^2 - n\bar{y}^2 - \hat{\beta}\, g$$

Unrestricted error S.S

$$= \sum_{l=1}^{n} (y_l - \bar{y})^2 - \sum_{i=1}^{p} \beta_i\, g_i$$

with $(n - p - 1)$ d.f.

$\therefore$ S.S due to hypothesis

$$= \Sigma_l\ (y_l - \bar{y})^2 - \hat{\beta}\ \Sigma_j\ g_j - \Sigma_l\ (y_l - \bar{y})^2 + \Sigma_i\ \hat{\beta}_i\, g_i$$

$$= \Sigma_i\ \hat{\beta}_i\, g_i - \hat{\beta}\ \Sigma_j\ g_j$$

$$= \Sigma_i\ \hat{\beta}_i\, g_i - \hat{\beta}\, g \text{ with } (p-1) \text{ d.f.}$$

Hence, the appropriate test is

$$F = \frac{\Sigma_i\ \hat{\beta}_i\, g_i - \hat{\beta}\, g/(p-1)}{\sum_{l=1}^{n} (y_l - \bar{y})^2 - \sum_{i=1}^{p} \beta_i\, g_i/(n-p-1)}$$

which is Snedecor's F with $(p-1)$ and $(n-p-1)$ d.f.

(iv) $H_0 : \beta_1 + \beta_2 + \ldots + \beta_p = C$

against $H_1 : \beta_1 + \beta_2 + \ldots + \beta_p \neq C$

Under H_0, $\hat{\beta}_1 + \hat{\beta}_2 + \ldots + \hat{\beta}_p - C = 0$

$$\text{Var}\,(\hat{\beta}_1 + \hat{\beta}_2 + \ldots + \hat{\beta}_p) = \Sigma_{i,j}\ s^{ij}\,\sigma^2, \text{ from (4)}$$

Also $\hat{\text{Var}}\,(\hat{\beta}_1 + \hat{\beta}_2 + \ldots + \hat{\beta}_p) = \Sigma_{i,j}\ s^{ij}\ \Sigma_l\ y_l^2 - n\bar{y}^2 - \Sigma_j\ \hat{\beta}_j\, g_j$

$$t = \frac{\hat{\beta}_1 + \hat{\beta}_2 + \ldots + \hat{\beta}_p - C}{\sqrt{\Sigma_{i,j}\ s_{ij}\ \Sigma_l\ y_l^2 - n\bar{y}^2 - \Sigma_j\ \hat{\beta}_j\, g_j/(n-p-1)}}$$

is distributed as Student's t with $(n-p-1)$ d.f. Alternatively,

$$F = t^2 = \frac{(\hat{\beta}_1 + \hat{\beta}_2 + \ldots + \hat{\beta}_p - C)^2/1}{\Sigma_{i,j}\ s^{ij}\ \Sigma_l\ y_l^2 - n\bar{y}^2 - \Sigma_j\ \hat{\beta}_j\, g_j/(n-p-1)}$$

is Snedecor's F with 1 and $(n-p-1)$ d.f.

(v) $H_0 : \beta_i = \beta_i^0$ **for some** $i = 1, 2, \ldots, p$

Under H_0, $E\,(\hat{\beta}_i) = \beta_i$ and $\text{Var}\,(\hat{\beta}_i) = s^{ii}\,\sigma^2$; from (1) and (3)

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$$\hat{\text{V}}\text{ar}\,(\hat{\beta}_i) = s^{ii}\,\hat{\sigma}^2$$

$$= s^{ii}\,\frac{\Sigma\,(y_l-\bar{y})^2 - \Sigma\,\hat{\beta}_i\,g_i}{(n-p-1)}\,, \quad \text{from (12)}$$

Thus
$$t = \frac{\hat{\beta}_i - \beta_i^0}{\sqrt{s^{ii}\,[\Sigma\,(y_l-\bar{y})^2 - \Sigma\,\hat{\beta}_i\,g_i]/(n-p-1)}}$$

is distributed as Student's t with $(n-p-1)$ d.f. Alternatively,

$$F = \frac{(\hat{\beta}_i - \beta_i^0)^2/1}{s^{ii}\,[\Sigma\,(y_l-\bar{y})^2 - \Sigma\,\hat{\beta}_i\,g_i]/(n-p-1)}$$

is Snedecor's F with 1 and $(n-p-1)$ d.f.

(vi) H_0 : A few coefficients in the multiple regression model are zero

We have the regression model

$$y_l = \alpha + \beta_1\,(x_{1l}-\bar{x}_1) + \ldots + \beta_p\,(x_{pl}-\bar{x}_p) + \epsilon_l$$

$$l = 1, 2, \ldots, n$$

Denoting $\hat{\beta}_i$ as the O.L.S.E. of β_i, $(i = 1, 2, \ldots, p)$, we have

$$\sum_{l=1}^{n}(y_l-\bar{y})^2 - \sum_{j=1}^{p}\beta_j\,g_j \sim \chi^2\,\sigma^2 \text{ with } (n-p-1) \text{ d.f., by (15)}$$

Let under H_0, only q of the partial regression coefficients $(q < p)$ are non zero. Then under H_0, the regression model is given by

$$y_l = \bar{\alpha} + \bar{\beta}_1\,(x_{1l}-\bar{x}_1) + \ldots + \bar{\beta}_q\,(x_{ql}-\bar{x}_q) + \epsilon_l$$

If $\hat{\bar{\beta}}_1, \hat{\bar{\beta}}_2, \ldots, \hat{\bar{\beta}}_q$ are the O.L.S.E. of $\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_q$ respectively, then $\sum_{l=1}^{n}(y_l-\bar{y})^2 - \sum_{j=1}^{q}\bar{\beta}_j\,g_j$ is the error sum of squares $\sim \chi^2\,\sigma^2$ with $(n-q-1)$ d.f.

Thus the S.S due to hypothesis = S.S due to error under H_0

$$= \left[\sum_{l=1}^{n}(y_l-\bar{y})^2 - \sum_{j=1}^{q}\bar{\beta}_j\,g_j\right] - \left[\sum_{l=1}^{n}(y_l-\bar{y})^2 - \sum_{j=1}^{p}\hat{\beta}_j\,g_j\right] \quad \leftarrow \text{unrestricted S.S}$$

$$= \sum_{j=1}^{p}\hat{\beta}_j\,g_j - \sum_{j=1}^{q}\hat{\bar{\beta}}_j\,g_j \sim \chi^2\,\sigma^2$$

with $(n-q-1)-(n-p-1) = (p-q)$ d.f.

Also unrestricted error S.S $= \sum_{l=1}^{n}(y_l-\bar{y})^2 - \sum_{j=1}^{p}\hat{\beta}_j\,g_j$

with $(n-p-1)$ d.f.

Thus
$$F = \frac{\sum_{j=1}^{p}\hat{\beta}_j\,g_j - \sum_{j=1}^{q}\hat{\bar{\beta}}_j\,g_j/(p-q)}{\sum_{l=1}^{n}(y_l-\bar{y})^2 - \sum_{j=1}^{p}\hat{\beta}_j\,g_j/(n-p-1)}$$

is distributed as Snedecor's F with $(p-q)$ and $(n-p-1)$ d.f.

(vii) Testing Equality of two regression lines

Suppose we have two regression lines given as follows:

$$y_l^{(1)} = \alpha^{(1)} + \beta_1^{(1)}\,(x_{1l}-\bar{x}_1^{(1)}) + \ldots + \beta_p^{(1)}\,(x_{pl}-\bar{x}_p^{(1)}) + \epsilon_l^{(1)} \qquad (16)$$

and
$$y_m^{(2)} = \alpha^{(2)} + \beta_1^{(2)}\,(x_{1m}-\bar{x}_1^{(2)}) + \ldots + \beta_p^{(2)}\,(x_{pm}-\bar{x}_p^{(2)}) + \epsilon_m^{(2)} \qquad (17)$$

OLS gives the two estimated regression equations as

$$y_l^{(1)} = \bar{y}^{(1)} + \hat{\beta}_1^{(1)}\,(x_{1l}-\bar{x}_1^{(1)}) + \ldots + \hat{\beta}_p^{(1)}\,(x_{pl}-\bar{x}_p^{(1)}) \qquad (18)$$

and

$$y_m^{(2)} = \bar{y}^{(2)} + \hat{\beta}_1^{(2)}\,(x_{1m}-\bar{x}_1^{(2)}) + \ldots + \hat{\beta}_p^{(2)}\,(x_{pm}-\bar{x}_p^{(1)}) \qquad (19)$$

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The normal equations corresponding to the first estimated regression is given by

$$g_i^{(1)} = \hat{\beta}_1^{(1)} s_{i1}^{(1)} + \ldots + \hat{\beta}_p^{(1)} s_{ip}^{(1)}$$

and the corresponding error S.S is (20)

$$\sum_{l=1}^{n_1} (y_l^{(1)} - \bar{y}^{(1)})^2 - \sum_{i=1}^{p} \hat{\beta}_i^{(1)} g_i^{(1)} \sim \chi^2 \sigma^2 \quad (21)$$

with $(n_1 - p - 1)$ d.f. Similarly, the normal equations corresponding to the second estimated regression is given by

$$g_i^{(2)} = \hat{\beta}_1^{(2)} s_{i1}^{(2)} + \ldots + \hat{\beta}_p^{(2)} s_{ip}^{(2)} \quad (22)$$

with the corresponding error S.S as

$$\sum_{m=1}^{n_2} (y_m^{(2)} - \bar{y}^{(2)})^2 - \sum_{i=1}^{p} \beta_i^{(2)} g_i^{(2)} \sim \chi^2 \sigma^2 \quad (23)$$

with $(n_2 - p - 1)$ d.f. Since the two S.S are independent

$$R_1^2 = \sum_{l=1}^{n_1} (y_l^{(1)} - \bar{y}^{(1)})^2 + \sum_{m=1}^{n_2} (y_m^{(2)} - \bar{y}^{(2)})^2 - \sum_{i=1}^{p} \hat{\beta}_i^{(1)} g_i^{(1)} - \sum_{i=1}^{p} \hat{\beta}_i^{(2)} g_i^{(2)} \sim \chi^2 \sigma^2 \quad (24)$$

with $(n_1 - p - 1 + n_2 - p - 1) = (n_1 + n_2 - 2p - 2)$ d.f.

Under the hypothesis of equality of two regression equations, it follows that $y_l^{(1)}$ and $y_m^{(2)}$, $(l = 1, 2, \ldots, n_1;\ m = 1, 2, \ldots n_2)$ belong to the same population. We shall denote the common regression by

$$Y_l = \alpha + \beta_1 (x_{1l} - \bar{x}_1) + \ldots + \beta_p (x_{pl} - \bar{x}_p) + \epsilon_l \quad (25)$$

where $l = 1, 2, \ldots n_1,\ (n_1 + 1), \ldots, (n_1 + n_2)$

While the first set of n_1 observations are employed to estimate the first regression equation (or the parameters of the same), the rest of them are employed to estimate the second regression equation. Also,

$$\bar{x}_j = \frac{n_1 \bar{x}_j^{(1)} + n_2 \bar{x}_j^{(2)}}{(n_1 + n_2)} \quad (26)$$

The O.L.S.E. of α is

$$\bar{Y} = \frac{n_1 \bar{Y}^{(1)} + n_2 \bar{Y}^{(2)}}{(n_1 + n_2)} \quad (27)$$

Further, the normal equations for obtaining the L.S. regression estimates

$\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_p$ are

$$g_i = \hat{\beta}_1 s_{i1} + \ldots + \hat{\beta}_p s_{ip},\ i = 1, 2, \ldots, p \quad (28)$$

where $$g_i = g_i^{(1)} + g_i^{(2)} = \frac{n_1 n_2}{(n_1 + n_2)} (\bar{y}^{(1)} - \bar{y}^{(2)})^2 \quad (29)$$

and $$s_{ij} = s_{ij}^{(1)} + s_{ij}^{(2)} = \frac{n_1 n_2}{(n_1 + n_2)} (\bar{x}_i^{(1)} - \bar{x}_i^{(2)})(\bar{x}_j^{(1)} - \bar{x}_j^{(2)}) \quad (30)$$

Under H_0, the error S.S is given by

$$R_0^2 = \sum_{l=1}^{n_1 + n_2} (y_l - \bar{y})^2 - \sum_{i=1}^{p} \hat{\beta}_i g_i \sim \chi^2 \sigma^2 \quad (31)$$

with $(n_1 + n_2 - p - 1)$ d.f.

Since (24) and (31) are independent,

$$R_0^2 - R_1^2 = \sum_{l=1}^{n_1 + n_2} (y_l - \bar{y})^2 - \sum_{i=1}^{p} \hat{\beta}_i g_i - \sum_{l=1}^{n_1} (y_l^{(1)} - \bar{y}^{(1)})^2 - \sum_{m=1}^{n_2} (y_m^{(2)} - \bar{y}^{(2)})^2 + \sum_{i=1}^{p} \hat{\beta}_i^{(1)} g_i^{(1)} + \sum_{i=1}^{p} \hat{\beta}_i^{(2)} g_i^{(2)}$$

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$$= \sum_{i=1}^{p} \hat{\beta}_i^{(1)} g_i^{(1)} + \sum_{i=1}^{p} \hat{\beta}_i^{(2)} g_i^{(2)}$$

$$+ \frac{n_1 n_2}{(n_1 + n_2)} (\bar{y}^{(1)} - \bar{y}^{(2)})^2 \sim \chi^2 \sigma^2 \qquad (32)$$

with $(p+1)$ d.f. under H_0

$$\therefore \quad \sum_{l=1}^{n_1+n_2} (y_l - \bar{y})^2 - \sum_{l=1}^{n_1} (y_l^{(1)} - \bar{y}^{(1)})^2 - \sum_{m=1}^{n_2} (y_m^{(2)} - \bar{y}^{(2)})^2$$

$$= \sum_{l=1}^{n_1+n_2} \left[y_l - \frac{n_1 \bar{y}^{(1)} + n_2 \bar{y}^{(2)}}{(n_1 + n_2)} \right]^2 - \sum_{l=1}^{n_1} (y_l^{(1)})^2 + n_1 (\bar{y}^{(1)})^2$$

$$- \sum_{m=1}^{n_2} (y_m^{(2)})^2 + n_2 (\bar{y}^{(2)})^2$$

$$= \sum_{l=1}^{n_1+n_2} y_l^2 - \frac{(n_1 \bar{y}^{(1)} + n_2 \bar{y}^{(2)})^2}{(n_1 + n_2)} - \sum_{l=1}^{n_1} (y_l^{(1)})^2 - \sum_{m=1}^{n_2} (y_m^{(2)})^2$$

$$+ n_1 (\bar{y}^{(1)})^2 + n_2 (\bar{y}^{(2)})^2$$

$$= \frac{-(n_1 \bar{y}^{(1)} + n_2 \bar{y}^{(2)})^2 + n_1 (n_1 + n_2) (\bar{y}^{(1)})^2 + n_2 (n_1 + n_2) (\bar{y}^{(2)})^2}{(n_1 + n_2)}$$

$$= \frac{-2 n_1 n_2 \bar{y}^{(1)} \bar{y}^{(2)} + n_1 n_2 (\bar{y}^{(1)})^2 + n_1 n_2 (\bar{y}^{(2)})^2}{(n_1 + n_2)}$$

$$= \frac{n_1 n_2}{(n_1 + n_2)} (\bar{y}^{(1)} - \bar{y}^{(2)})^2$$

Now from (24) and (32)

$$F = \frac{(R_0^2 - R_1^2)/(p+1)}{R_1^2/(n_1 + n_2 - 2p - 2)}$$

$$= \frac{(R_0^2 - R_1^2)(n_1 + n_2 - 2p - 2)}{(p+1) R_1^2}$$

is distributed as Snedecor's F with $(p+1)$ and $(n_1 + n_2 - 2p - 2)$ degrees of freedom.

Remarks : The above testing procedure is adopted when we require to test the hypothesis

$$H_0 : a^{(1)} = a^{(2)}, \; b_1^{(1)} = b_1^{(2)}, \ldots, b_p^{(1)} = b_p^{(2)}$$

However, when the hypothesis is simply

$$H_0' : \beta_1^{(1)} = \beta_1^{(2)}, \ldots, \beta_p^{(1)} = \beta_p^{(2)}$$

i.e. the intercepts of the two regression equations are not tested, the normal equations for estimating the regression coefficients are obtained by adding (20) and (22). i.e.,

$$g_i^{(1)} + g_i^{(2)} = \hat{\beta}_1 (s_{i1}^{(1)} + s_{i1}^{(2)}) + \ldots + \hat{\beta}_p (s_{ip}^{(1)} + s_{ip}^{(2)})$$

S.S due to estimate under H_0' is

$$\sum_{i=1}^{p} \hat{\beta}_1 (g_i^{(1)} + g_i^{(2)})$$

Thus under H_0', we have

$$\sum_{l=1}^{n_1} (y_l^{(1)} - \bar{y}^{(1)})^2 + \sum_{m=1}^{n_2} (y_m^{(2)} - \bar{y}^{(2)})^2 - \sum_{i=1}^{p} \hat{\beta}_1 (g_i^{(1)} + g_i^{(2)}) = R_2^2 \qquad (33)$$

$$R_2^2 \sim \chi^2 \sigma^2 \text{ with } (n_1 + n_2 - p - 2) \text{ d.f.}$$

Also from (24)

$$R_1^2 \sim \chi^2 \sigma^2 \text{ with } (n_1 + n_2 - 2p - 2) \text{ d.f.}$$

$$\therefore \quad R_2^2 - R_1^2 \sim \chi^2 \sigma^2$$

with $(n_1 + n_2 - p - 2) - (n_1 + n_2 - 2p - 2) = p$ d.f.

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Hence, $$F = \frac{R_2^2 - R_1^2/p}{R_1^2/(n_1 + n_2 - 2p - 2)}$$

is the appropriate statistic for testing H_0' which is distributed as Snedecor's F with p and $(n_1 + n_2 - 2p - 2)$ d.f.

Problem 5 : R.A. Fisher gives the following data for the effect of longitude (X_1), latitude (X_2), and altitude (X_3) in yield (Y) based on a sample size 186. Denoting the small letters as deviation from the means of the corresponding variables, we have

$\Sigma y^2 = 1786.6$ $\quad \Sigma x_1^2 = 1934.1$

$\Sigma x_1 y = 1137.4$ $\quad \Sigma x_1 x_2 = -772.2$

$\Sigma x_2 y = 592.9$ $\quad \Sigma x_2^2 = 2889.5$

$\Sigma x_3 y = 891.8$ $\quad \Sigma x_1 x_3 = 924.1$

$\Sigma x_3^2 = 1750.8$ $\quad \Sigma x_2 x_3 = 119.6$

(*i*) Estimate the partial regression coefficients in the data and test their significances.

(*ii*) Test the equality of regression coefficients.

(*iii*) Test $\beta_1 = 0$ and

(*iv*) $\beta_1 + \beta_2 + \beta_3 = 0.25$

Let the line of regression of Y on X_1, X_2 and X_3 is given by

$$y = b_1 x_1 + b_2 x_2 + b_3 x_3$$

The constant has been omitted since the variables have been measured from their respective means. The problem is to minimise Residual S.S., i.e. $\Sigma (y - b_1 x_1 - b_2 x_2 - b_3 x_3)^2$.

The normal equations are:

$$b_1 \Sigma x_1^2 + b_2 \Sigma x_1 x_2 + b_3 \Sigma x_1 x_3 = \Sigma x_1 y$$

$$b_1 \Sigma x_1 x_2 + b_2 \Sigma x_2^2 + b_3 \Sigma x_2 x_3 = \Sigma x_2 y$$

$$b_1 \Sigma x_1 x_3 + b_2 \Sigma x_2 x_3 + b_3 \Sigma x_3^2 = \Sigma x_3 y$$

$$\begin{pmatrix} \Sigma x_1^2 & \Sigma x_1 x_2 & \Sigma x_1 x_3 \\ \Sigma x_1 x_2 & \Sigma x_2^2 & \Sigma x_2 x_3 \\ \Sigma x_1 x_3 & \Sigma x_2 x_3 & \Sigma x_3^2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} \Sigma x_1 y \\ \Sigma x_2 y \\ \Sigma x_3 y \end{pmatrix}$$

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} \Sigma x_1^2 & \Sigma x_1 x_2 & \Sigma x_1 x_3 \\ \Sigma x_1 x_2 & \Sigma x_2^2 & \Sigma x_2 x_3 \\ \Sigma x_1 x_3 & \Sigma x_2 x_3 & \Sigma x_3^2 \end{pmatrix}^{-1} \begin{pmatrix} \Sigma x_1 y \\ \Sigma x_2 y \\ \Sigma x_3 y \end{pmatrix}$$

Thus, we can either find out the inverse of the S.P. matrix and then postmultiply the same with the column vector of $\Sigma x_1 y$, $\Sigma x_2 y$, and $\Sigma x_3 y$ or we can obtain the estimates b_1, b_2, b_3 directly by pivotal condensation shown in Table-1. The pivotal row (30) gives the solution for b_3 and by substituting the value of b_3 and b_2 in rows (20) and (10) respectively, row (10′) is obtained. Thus, b_3, b_2 and b_1 are given by

$$\begin{pmatrix} b_3 \\ b_2 \\ b_1 \end{pmatrix} = \begin{pmatrix} .1234 \\ .3822 \\ .6817 \end{pmatrix}$$

Table 1 : Method of Pivotal Condensation for the Determination of Inverse of S.P. Matrix and the Solution of Normal Equations in Connection with the Estimation of Partial Regression Coefficients

x_1	x_2 (1)	x_3	$\Sigma x_i y_i$ (2)	Unit Matrix for the Inverse (3)			(1) + (3)
1934.1	− 772.2	924.1	1137.4	1	0	0	2087.0
−772.2	2889.5	119.6	592.9	0	1	0	2237.9
924.1	119.6	1750.8	391.8	0	0	1	2795.5
1	− .39926	.47779	.58808	.00052	0	0	1.079054
0	2581.2	488.55	1097.0	.39923	1	0	
0	488.55	1309.28	34.35	− .47776	0	1	1798.346
	1	.18927	.11056	.000155	.000387	0	1.189815
	0	1216.8	150.2	− .55349	− .18907	1	1217.06
0	0	1	.1234	− .00045	− .00016	.000822	1.00023
	1	0	.3822	.00024	.00042	− .00016	1.0005
	0	0	.6817	.00083	.00024	− .000455	1.0006

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Here $b_1 = .6817$, $b_2 = .3822$ and $b_3 = .1234$, taking the last entries in column (2) in reverse order. Similarly, inverse of the SP Matrix

$$\begin{bmatrix} .00083 & .00024 & -.000455 \\ .00024 & .00042 & -.00016 \\ -.00045 & -.00016 & .000822 \end{bmatrix}$$

is obtainable from Column (3) while taking the matrix in reverse order.

Alternatively, the inverse of the S.P. matrix by pivotal condensation as seen from Table-1 is

$$\begin{pmatrix} .000831 & .000240 & -.000455 \\ .000241 & .000416 & -.000156 \\ -.000455 & .000155 & .000822 \end{pmatrix}$$

This matrix when postmultiplied by

$$\begin{pmatrix} \Sigma x_1 y \\ \Sigma x_2 y \\ \Sigma x_2 y \end{pmatrix} = \begin{pmatrix} 1137.4 \\ 592.9 \\ 391.8 \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}$$

gives

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} .6817 \\ .3822 \\ .1234 \end{pmatrix}$$

which is same as the estimates obtained earlier directly.

To test $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$, we have the minimised residual S.S. = 1186.6. The reduction in the above sum of squares is due to regression. The analysis of the S.S. is given in ANOVA (Table-2).

Table-2 : ANOVA

Source of Variation	d.f.	S.S.	M.S.	F
Regression	3	488.01386	162.6713	42.3785
Residual	182	698.58614	3.8385	
Total	185	1186.60000		

The variance ratio $F = 42.3785$ with 3 and 182 d.f. is thus significant (*i.e.* $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$, is discredited) at 99% level of significance.

The multiple correlation R is given by

$$R^2 = \frac{b_1 Q_1 + b_2 Q_2 + b_3 Q_3}{\text{T.S.S.}}$$

$$= \frac{(.6817)\ (1137.4) + (.3820)\ (529.9) + (.1234)\ (891.8)}{1186.6}$$

$$= 0.91677$$

(ii) Testing the equality of regression coefficients

If b is the least squares estimate of the common value of the regression coefficients, β_1, β_2, β_3, then

$$b = \frac{Q_1 + Q_2 + Q_3}{\sum_j s_{ij}}$$

$$= \frac{1137.4 + 592.9 + 891.8}{1934.1 - 2\,(772.2) + 2\,(119.6) + 1750.8 + 2889.5}$$

$$= 0.3684$$

The S.S. due to $b = (.3684)\ (Q_1 + Q_2 + Q_3) = 965.98164$

Table 3 : ANOVA

Source of Variation	d.f.	S.S.	M.S.	F
Due to b	1	965.9816	965.9816	
Deviation from equality of regressiion coeffts.	2	121.8538	60.9269	110.4235
Due to b_1, b_2, b_3	3	1087.8355	362.6118	
Error	179	98.7645	0.5518	
Total	185	1186.6		

Conclusion : H_0 is discredited

(iii) To test $\beta_1 = 0$

We have

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$$S^{ij} = (S_{ij})^{-1} = \begin{pmatrix} .000831 & .000240 & -.000455 \\ .000240 & .000416 & -.000156 \\ -.000455 & -.000156 & .000822 \end{pmatrix}$$

$$\text{Var}(b_1) = .000831\,\sigma^2$$

$$= .000831 \times 3.8384 = 0.00319$$

$$b_1 = 0.6817$$

$$|t| = \frac{0.6817}{\sqrt{.000831 \times 3.8384}} = 15.14$$

with $(185 - 3) = 182$ d.f. Since sample size is very large this result can be compared by the standardised normal deviates i.e., a value of $t > 1.96$ is considered to be significant at 5% level of significance.

(iv) **To test $\beta_1 + \beta_2 + \beta_3 = 0.25$**

We have

$$\text{Var}(b_1 + b_2 + b_3) = \Sigma\, s^{ij}\sigma^2$$

$$= [.000831 + 2\,(.00024) - 2\,(.000455) + .000416 - 2\,(.000156) + .000822]\,3.8384$$

$$= .00509$$

Hence, under H_0

$$t = \frac{b_1 + b_2 + b_3 - 0.25}{\sqrt{.00509}} = 13.14$$

is distributed as Student's t with 182 d.f. As before, since the sample size is very large, this test is equivalent to standard normal deviate and hence, H_0 is rejected.

Problem 6 : Given $n = 40$ $\Sigma x_0 = 159.7$ $\Sigma x_1 = 211$ $\Sigma x_0^2 = 846.75$ $\Sigma x_0 x_1 = 10766.95$ $\Sigma x_1^2 = 144461$ $S_{00} = 209.148$ $S_{01} = 2338.733$ $S_{11} = 33052.975$ $a = 0.25830$ $b = 0.070757$ where $x_0 = a + bx_1$. Examine which of the regression of weight of dry jute fibre (x_0) on the weight of the green plant (x_1) can be taken as

(i) $X_0 = 0.25 + 0.1\,X_1$

(ii) $X_0 = CX_1$, where C is some suitable constant.

S.S. due to departure from the line of regression

$X_0 = .25830 + 0.070757\,X_1$

is given by

$$R_{00} = S_{00} - bS_{01}$$

$$= 209.148 - (.070757)\,(2338.733)$$

$$= 43.6663$$

with $(n - 2) = (40 - 2) = 38$ d.f.

Now considering the first regression equation

$X_0 = 0.25 + 0.1\,X_1$

S.S. due to departure from linearity

$$= \Sigma\,(X_0 - 0.25 - 0.1\,X_1)^2$$

$$= \Sigma\,X_0^2 + 0.0625 + 0.01\,\Sigma\,X_1^2 - 0.50\,\Sigma\,X_0 + 0.05\,\Sigma\,X_1 - 0.02\,\Sigma\,X_0 X_1$$

$$= 846.75 + .0625 + .01\,(144461) - .5\,(159.7) + .05\,(2111) - 0.2\,(10766.95)$$

$$= 166.17$$

To test whether $X_0 = 0.25 + 0.1\,X_1$ can be taken as the regression equation, we take

$$F = \frac{(166.17 - 43.6663)/2}{43.6663/38}$$

$$= 53.304$$

which is a variance ratio with 2 and 38 d.f. and is highly significant. Hence, $x_0 = 0.25 + 0.1\,X_1$ cannot be accepted as the line of regression.

Next consider the suitability of $X_0 = CX_1$ as the regression equation. This means that whether by dropping α from $X_0 = \alpha + \beta X_1$ it can be statistically accepted as regression equation. We test the hypothesis, $H_0 : \alpha = 0$.

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Now consider the least squares regression of X_0 on X_1, i.e. $X_0 = a + bX_1$

Let S_0 = Standard Error of α

Then $$S_0 = \sqrt{\frac{[1/n + \bar{x}_1/S_{11}]^2}{R_{00}/(n-2)}}$$

$$= 0.3543$$

The statistic t obtained to test $\alpha = 0$ is given by

$$t = (a - 0)/S_0 = .25830/.3543$$

$$= 0.724$$

with $(40 - 2) = 38$ d.f. is not significant. Hence the regression equation $X_0 = CX_1$ for some suitable C is tenable.

Problem 7 : Testing the linearity of regression.

The following table gives the frequency distribution of yields of dry bark in ounce (y) on the age in years (x) of 156 plants. Obtain the linear regression of y on x and the coefficient of correlation between y and x. Examine the linearity of regression.

Age in years	Yield in OZS					
	4–7	6–11	12–18	16–19	20–23	24–27
3 –4	2	—	—	—	—	—
5 – 6	3	6	6	1	—	—
7 – 8	3	7	10	5	—	—
9 – 10	—	8	15	10	10	—
11 – 12	—	—	12	19	15	5
13 – 14	—	—	2	4	10	4

The regression of y on x is

$$y = 2.4576 + 1.3762x$$

The coefficient of correlation

$$r = 0.6636$$

The correlation ratio

$$\eta = 0.6687$$

To test the linearity of regression, we have

$$F = \frac{(\eta^2 - r^2)/(k-2)}{(1-\eta^2)/(n-k)}$$

where $n = 157$ and $k = 6$

$\Rightarrow$ $F = 0.464$ with 4 and 15 d.f.

This is not significant at 5% level which shows that non-linearity assumption cannot be ruled out.

Problem 8 : Confidence limits for regression and correlation coefficients

The distribution of the prices of potatoes per 100 kg in 1986 (x) and 1987 (y) based on the quotation of $n = 46$ towns give the following estimates.

$\bar{x} = 660.57$ $\bar{y} = 732.59$

$s_1 = 106.8$ $s_2 = 130.91$

$r = 0.7928$ $b_{yx} = 0.7007$ and $b_{xy} = 0.8971$

Obtain the confidence limits for the regression coefficient of Y on X and the population correlation coefficient.

95% confidence limits for the regression coefficient is given by

$$b_{yx} \pm t_{.05}\left[\frac{(n-2)\Sigma(x_i - \bar{x})^2}{\Sigma(y_i - Y_i)^2}\right]^{1/2}$$

and $\Sigma(y_i - Y_i)^2$ = S.S. due to regression of Y on X

$$= NS_2^2(1 - r^2)$$

$$\Sigma(y_i - Y_i)^2 = 46\,(120.9)^2\,[1 - (0.7928)^2]$$

$$= 249807.81$$

$$(n-2)\Sigma(x_i - \bar{x})^2 = 106.8 \times 46\,(46 - 2)$$

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$$= 216163.2$$

$$\frac{(n-2)\Sigma (x_i - \bar{x})^2}{\Sigma (y_i - Y_i)^2} = 0.8652$$

$$\left[\frac{(n-2)\Sigma (x_i - \bar{x})^2}{\Sigma (y_i - Y_i)^2}\right]^{-\frac{1}{2}} = 1/\sqrt{0.8652} = 1.075$$

confidence limits of b_{yx} is given by $0.7007 \pm 2.02\,(1.0750)$

$$= -1.47008,\ 2.8723$$

i.e., $P[-1.47008 \le b_{yx} \le 2.8723] = 0.95$

Confidence limits for the Correlation Coefficient

Since the distribution of r (when $\rho \neq 0$) is markedly skew, even under large sample approximation, it significantly deviates from normality. Therefore, we employ the transformation

$$Z = \frac{1}{2}\log_e \frac{1+r}{1-r}$$

which is approximately normally distributed with standard deviation

$$\frac{1}{\sqrt{(n-3)}}, \text{ Then } \pm t = \frac{(Z-s)}{S_z}$$

where S_z = S.E. (Z) and $s = E(Z)$, is distributed as standard normal deviate (or as Student's t with infinite d.f.) giving the upper and lower confidence limits l_1 and l_2 as

$$l_1 = Z - t_{0.05}\, S_z$$

$$l_2 = Z + t_{0.05}\, S_z$$

and on transforming l_1 and l_2 back to value of r, we get the confidence limits at 5% point. The value of t in these tests arises from assuming the d.f. to be infinite. Accordingly, the table of normal deviate will give the same result as the table of t. Following this procedure we get

$$Z = \frac{1}{2}\log_e \frac{1+0.7928}{1-0.7928}$$

$$= 1.07893$$

and

$$S_z = \frac{1}{\sqrt{46.3}} = 0.1525$$

$$l_1 = Z - t_{0.05}\, S_z$$

$$= 1.07893 - 1.96 \times 0.152499$$

$$= 0.78003$$

Also if r_l = lower limit of r corresponding to l_1

l_1 = lower limit of z,

then $$l_1 = \frac{1}{2}\log_e \frac{1+r_l}{1-r_l}$$

or $$0.78003 = \frac{1}{2}\log_e \frac{1+r_l}{1-r_l}\, 2.30259$$

$\Rightarrow$ $$r_l = 0.65272$$

Similarly, $$l_2 = Z + t_{0.05}\, S_z$$

$$= 1.07893 + 1.96 \times 0.152499$$

$$= 1.37783$$

$$= \frac{1}{2}\log_e \frac{1+r_u}{1-r_u}$$

where r_u = upper confidence limit of r corresponding to l_2, viz., the upper confidence limit of z

or $$2.75566 = \log_e \frac{1+r_u}{1-r_u}\, 2.30259$$

$\Rightarrow$ $$r_u = 0.88046$$

Thus, $$Pr[0.65272 \le \rho \le 0.88046 \mid r] = 0.95$$

where $$r = 0.7928.$$

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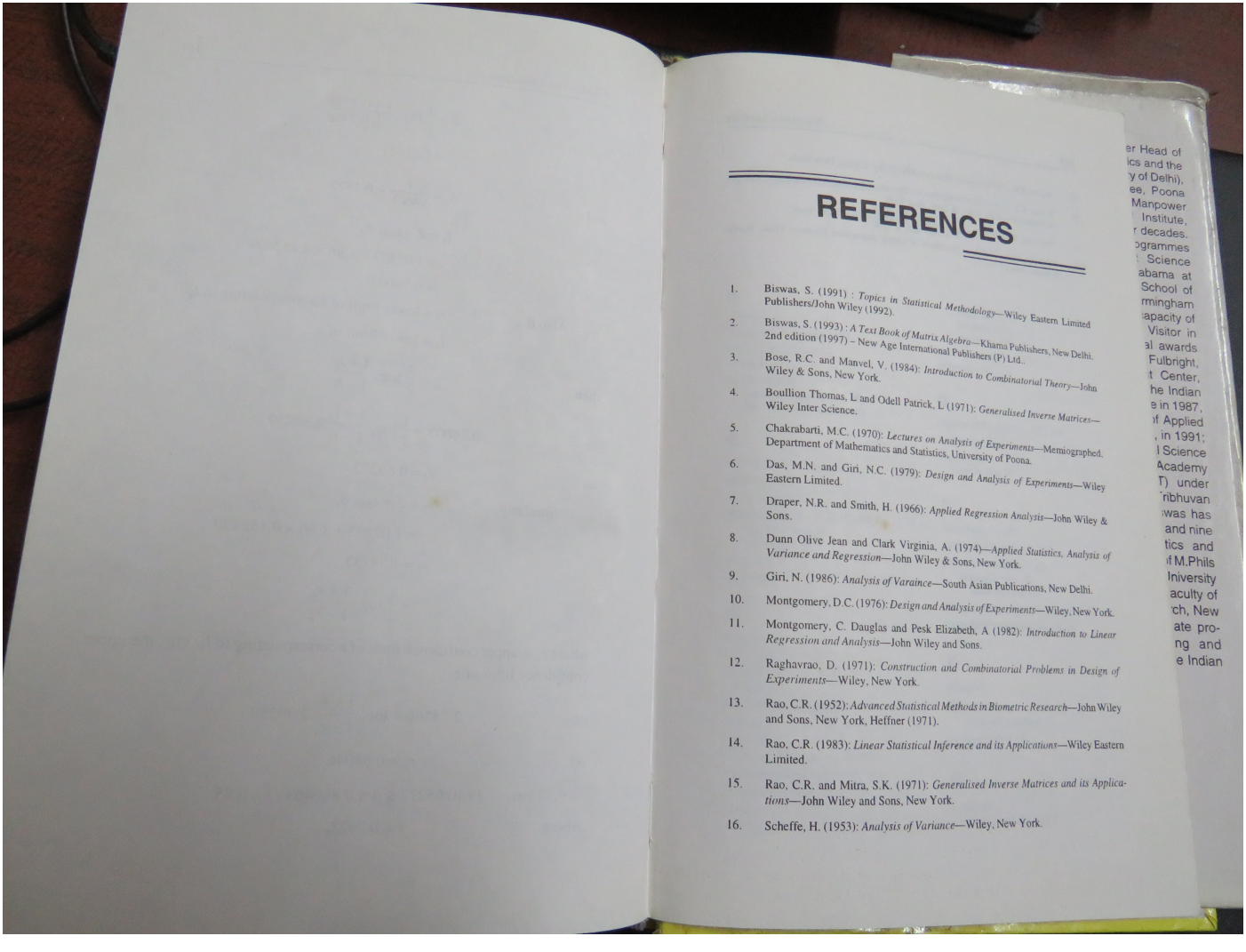

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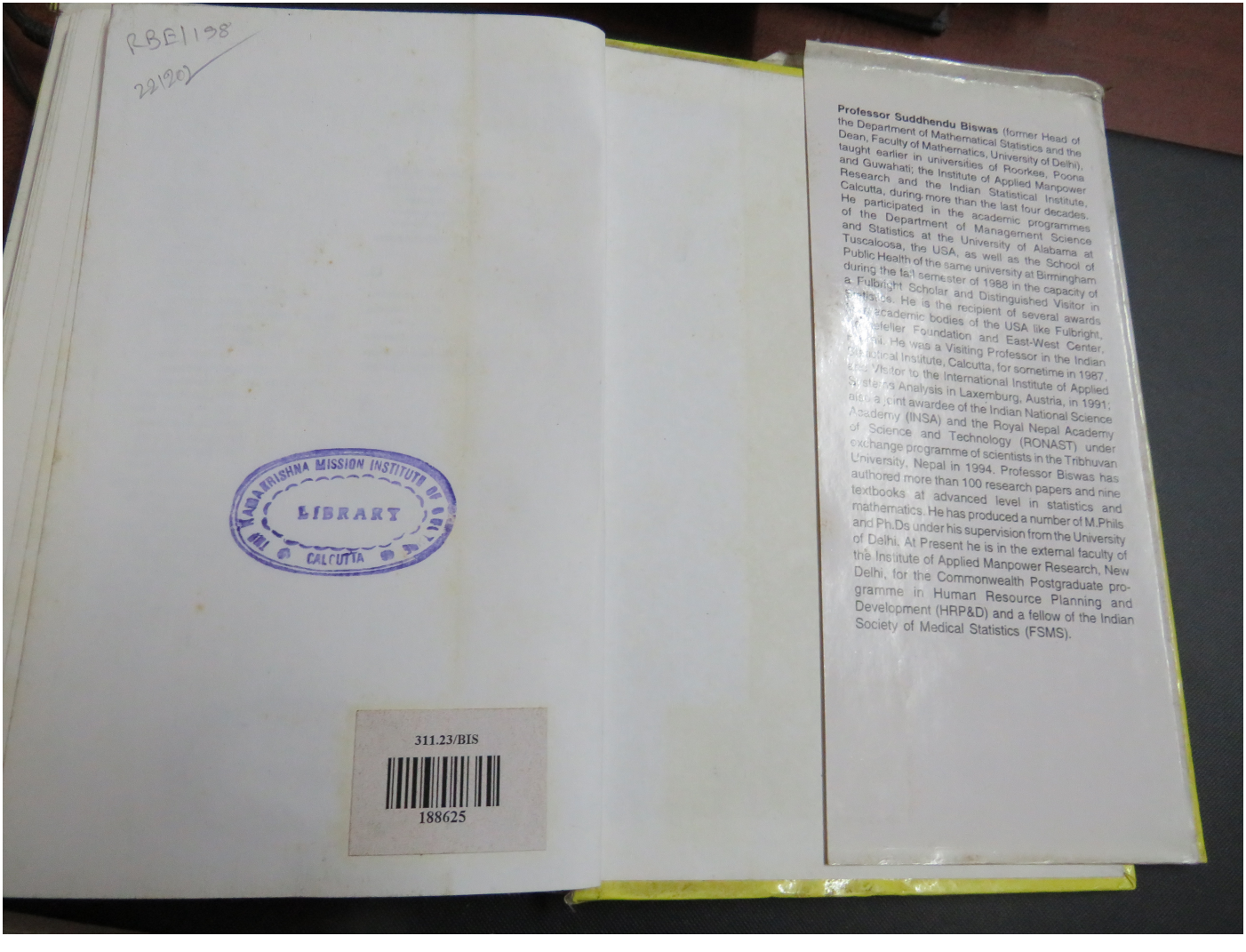

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